

Übungsblatt 4 - Musterlösung

1 a) $f(x) = ax + b$

$$F(x) = \frac{1}{2} ax^2 + bx + c$$

b) $f(x) = (ax + b)^{-1}$

$$F(x) = \frac{1}{a} \ln(ax + b)$$

c) $f(x) = x^{-5}$

$$F(x) = -\frac{1}{4} x^{-4}$$

d) $f(x) = \sin(3x)$

$$F(x) = -\frac{1}{3} \cos(3x)$$

e) $f(x) = (x+1)^{1/2}$

$$F(x) = \frac{2}{3} (x+1)^{3/2}$$

f) $f(x) = (a+x)^{-1/2}$

$$F(x) = -2(a+x)^{1/2}$$

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$$a) F(x) = \frac{1}{3} x^3$$

$$\int_a^b dx x^2 = \left[\frac{1}{3} x^3 \right]_a^b = \underline{\underline{\frac{1}{3} (b^3 - a^3)}}$$

$$b) F(x) = \ln x$$

$$\int_a^b dx \frac{1}{x} = \ln b - \ln a = \underline{\underline{\ln \frac{b}{a}}}$$

$$c) F(x) = \frac{1}{5} x^5 + \frac{2}{3} x^3 + x$$

$$\int_0^3 dx (x^2 + 1)^2 = 3 \left(\frac{81}{5} + 6 + 1 \right)$$

$$d) F(x) = -e^{-x}$$

$$\int_0^a dx e^{-x} = e^{-a} - e^0 = \underline{\underline{e^{-a} - 1}}$$

$$e) \int_a^b dx \frac{1}{\sqrt{(x-a)(b-x)}} \left[\begin{array}{l} \text{Substitution} \\ z = x - \underbrace{\frac{(a+b)}{2}}_{=c/2} \end{array} \right]$$

$$\int_{-c/2}^{c/2} dz \frac{1}{\sqrt{-z^2 + \frac{c^2}{4}}}$$

$$\left[\begin{array}{l} \text{Substitution} \\ \xi = \frac{2}{c} z \end{array} \right]$$

$$\frac{c^2}{4} \int_{-1}^1 d\xi \frac{1}{\underbrace{\sqrt{-\xi^2 + 1}}_{f(\xi)}} \rightarrow F(\xi) = \arcsin(\xi)$$

$$\begin{aligned} \int_a^b dx \frac{1}{\sqrt{(x-a)(b-x)}} &= \frac{(b-a)^2}{4} \left(\arcsin(1) - \arcsin(-1) \right) \\ &= \frac{(b-a)^2}{4} \cdot \pi \end{aligned}$$

$\begin{matrix} \pi/2 & & -\pi/2 \\ \uparrow & & \uparrow \end{matrix}$

$$f) \int_0^{2\pi} dx \cos^2 x \stackrel{\substack{\uparrow \\ \text{Symmetrie}}}{=} \int_0^{2\pi} dx \sin^2 x$$

$$\Rightarrow 2 \int_0^{2\pi} dx \cos^2 x = \int_0^{2\pi} (\underbrace{\cos^2 x + \sin^2 x}_{=1}) dx = \int_0^{2\pi} dx$$

$$\Rightarrow 2 \int_0^{2\pi} dx \cos^2 x = 2\pi \quad ; \quad \int_0^{2\pi} dx \cos^2 x = \underline{\underline{\pi}}$$

A3

$$a) \int \frac{x^2}{x^3+1} dx$$

Subst.
 $u = x^3 + 1$
 $\frac{du}{dx} = 3x^2$

$$\begin{aligned} \int \frac{1}{3} \frac{1}{u} du &= \frac{1}{3} \ln u + C \\ &= \frac{1}{3} \ln(x^3 + 1) + C \end{aligned}$$

$$b) \int \frac{\cos x}{\sin x} dx$$

$u = \sin x$
 $\frac{du}{dx} = \cos x$

$$\begin{aligned} \int \frac{1}{u} du &= \ln u + C \\ &= \ln(\sin x) + C \end{aligned}$$

$$c) \int e^{\cos x} \sin x dx$$

$u = \cos x$
 $\frac{du}{dx} = -\sin x$

$$\begin{aligned} -\int e^u du &= -e^u + C \\ &= -e^{\cos x} + C \end{aligned}$$

$$d) \int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

$$\boxed{\begin{array}{l} u = \cosh x \\ \frac{du}{dx} = \sinh x \end{array}}$$

$$= \int \frac{1}{u} \, du = \ln u + C$$

$$= \ln(\cosh x) + C$$

$$e) \int \frac{1}{x \ln^2 x} \, dx$$

$$\boxed{\begin{array}{l} \ln x = u, \quad x = e^u \\ \frac{du}{dx} = \frac{1}{x} \end{array}}$$

$$\int \frac{1}{u^2} \, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$f) \int x e^{-ax^2} \, dx$$

$$\boxed{\begin{array}{l} u = ax^2 \\ \frac{du}{dx} = 2ax \end{array}}$$

$$= \int \frac{1}{2a} e^{-u} \, du = -\frac{1}{2a} e^{-u} + C$$

$$= -\frac{1}{2a} e^{-ax^2} + C$$

4 a) $\int \frac{1}{\sin x} dx$

$$\boxed{\begin{aligned} t &= \tan \frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{\cos^2 \frac{x}{2}} \end{aligned}}$$

$$\begin{aligned} &\int \frac{\cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dt = \frac{1}{2} \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} dt = \frac{1}{2} \int \frac{1}{t} dt \\ &\stackrel{\sin x}{=} \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx = \frac{1}{2} \ln(t) + C = \frac{1}{2} \ln\left(\tan \frac{x}{2}\right) + C \end{aligned}$$

b) $\int \frac{1}{1 + \sin^2 x} dx$

$$\boxed{\begin{aligned} t &= \sqrt{2} \tan x \\ \frac{dt}{dx} &= \sqrt{2} \frac{1}{\cos^2 x} \end{aligned}}$$

$$\stackrel{\frac{1}{\sqrt{2}}}{=} \int dt \frac{\cos^2 x}{2 \sin^2 x + \cos^2 x}$$

$$1 = \sin^2 x + \cos^2 x$$

$$= \frac{1}{\sqrt{2}} \int dt \frac{1}{2 \underbrace{\frac{\sin^2 x}{\cos^2 x}}_{t^2} + 1} = \sqrt{2} \int dt \frac{1}{t^2 + 1}$$

$$= \frac{1}{\sqrt{2}} \arctan(t) + C$$

$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C$$