10 Fine-structure and time-dependent perturbation theory

Exercises graded according to correctness:

Aufgabe 10.1: Clebsch-Gordan coefficients for $l = 1$ and $s = 1/2$

Consider a rotator with spin $s = 1/2$ and angular momentum $l = 1$. The corresponding Hilbert space is six-dimensional and spanned by the simultaneous eigenfunctions $|l, m, s, m_s\rangle$ of $\hat{l}^2$, $\hat{l}_z$, $\hat{s}^2$ and $\hat{s}_z$, where $l = 1$, $m = -1, 0, 1$, $s = 1/2$ and $m_s = -1/2, 1/2$. The goal of this exercise is to construct from these states the eigenstates of the operators $\hat{\boldsymbol{j}}^2 = \hat{\boldsymbol{l}}^2 + \hat{\boldsymbol{s}}^2$ and $\hat{j}_z$, where $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$ is the total angular momentum.

(a) A given state $|\psi\rangle$ of the rotator can be expanded in the six basis states,

$$|\psi\rangle = \sum_{m=-1}^{1} \sum_{m_s=-1/2}^{1/2} a_{m,m_s} |l, m, s, m_s\rangle,$$

where a_{m,m_s} are complex numbers, $m = -1, 0, 1$, $m_s = -1/2, 1/2$, and $l = 1$, $s = 1/2$. This state can be represented as a coulmn vector

$$|\psi\rangle = \begin{pmatrix} a_{-1,-1/2} \\ a_{-1,1/2} \\ a_{0,-1/2} \\ a_{0,1/2} \\ a_{1,-1/2} \\ a_{1,1/2} \end{pmatrix}$$

In this notation operators are represented by 6×6 matrices. What is the matrix representation of the operator $\hat{j}_z$? What are the eigenvalues and their degeneracy? Give the corresponding eigenvectors.

(b) Show that $\hat{\boldsymbol{j}}^2 = \hat{\boldsymbol{l}}^2 + \hat{\boldsymbol{s}}^2 + \hat{l}_+ \hat{s}_- + \hat{l}_- \hat{s}_+ + 2\hat{l}_z \hat{s}_z$.

(c) Find the matrix representation of the operator $\hat{\boldsymbol{j}}^2$.
Hint: Use that $\hat{\boldsymbol{j}}^2 = \hat{\boldsymbol{l}}^2 + \hat{\boldsymbol{s}}^2 + 2\hat{l}_z \hat{s}_z + \hat{l}_+ \hat{s}_- + \hat{l}_- \hat{s}_+$.

(d) Since operators $\hat{\boldsymbol{j}}^2$ and $\hat{j}_z$ commute they share a basis of common eigenvectors $|j, m_j, l, s\rangle$ (with $s = 1/2$). Find these basis vectors.

Remark: the scalar products $\langle l, m, s, m_s | j, m_j, l, s\rangle$, which determine the coefficients in the expansion of the eigenvectors $|j, m_j, l, s\rangle$ in the basis of the product states $|l, m, s, m_s\rangle$, are called "Clebsch-Gordan coefficients".

Exercises graded according to efforts:

Aufgabe 10.2: Normal and anomalous Zeeman effect

The energy levels of the hydrogen atom are split by a magnetic field. The form of the splitting depends on the relative magnitude of the magnetic (Zeemann) energy and the fine structure splitting. In case of the "normal Zeeman effect" (or "Paschen-Back effect") the magnetic energy is larger than the fine structure splitting. To account for this effect one first considers the magnetic energy and then accounts for possible additional relativistic corrections in perturbation theory.

(a) The Hamilton operator of the hydrogen atom in a homogeneous magnetic field in z-direction is

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} - \frac{e^2}{r} - \frac{\mu_{\mathrm{B}}}{\hbar} B_z(\hat{l}_z + g\hat{s}_z),$$

where $g = 2$. Show that the states $|nlmm_s\rangle$ are eigenstates of this Hamilton operator and calculate the corresponding energy eigenvalues. Describe the splitting of the energy levels $n = 1$ und $n = 2$. Is the degeneracy of the second level $n = 2$ fully lifted?

(b) In the limit of the normal Zeeman effect relativistic corrections are small compared to the magnetic field induced splitting. One can therefore treat the effect of relativistic corrections in perturbation theory. Calculate additional shifts or splittings of the energy levels resulting from the spin orbit coupling,

$$\hat{H}_{\mathrm{so}} = \frac{1}{2m^2c^2} \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{l}} \left(\frac{1}{r} \frac{d}{dr} V(r) \right),$$

with $V(r) = -e^2/r$.

1. Hint: Use the identity $\hat{\boldsymbol{l}} \cdot \hat{\boldsymbol{s}} = \frac{1}{2}(\hat{l}_+\hat{s}_- + \hat{l}_-\hat{s}_+) + \hat{l}_z\hat{s}_z$. 2. Hint: Use that

$$\langle nlmm_s | r^{-3} | nlmm_s\rangle = \frac{2Z^3}{l(l+1)(2l+1)n^3a_0^3}$$

for $l > 0$.

In case of the "anomalous Zeeman effect" the magnetic energy is much smaller than the fine structure splitting. In this case one first considers the fine-structure splitting. Afterwards, additional splittings induced by the magnetic field can be accounted for by perturbation theory.

(c) Taking into account relativistic corrections, bound eigenstates $|njlm_j\rangle$ of the hydrogen atom are labeled by the quantum numbers n, j, l, und m_j, where $j = l \pm 1/2$ and

$$E_{nj} = -\frac{e^2}{2a_0 n^2}\left[1 - \frac{\alpha^2}{n^2}\left(\frac{3}{4} - \frac{n}{j+1/2}\right)\right].$$

Calculate the additional energy spitting due to the Zeeman-effect for the case $l = 1$. *Hint: Use your solution of exercise* **??**.

Aufgabe 10.3: Harmonic oscillator in a time-dependent electric field

Consider a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$.

(a) What is the Hamilton operator for such system?

At time $t = 0$ let the oscillator be in the energy eigenstate $|n\rangle$. Calculate the probability, that at a time $t \to \infty$ the oscillator is in a different energy eigenstate $|n'\rangle$ for the cases that

(b) $E(t) = E_0 e^{-\gamma t}$ (mit $\gamma > 0$),

(c) $E(t) = E_0 \gamma^{-1} \delta(t - t_0)$ (mit $t_0 > 0$).