2 Complex Vector Spaces

Exercises graded according to correctness:

Problem 2.1: Complex vector space

Let us consider the following vector in a two-dimensional vector space

$$\boldsymbol{e} = c \begin{pmatrix} i \\ 1 \end{pmatrix}$$

(a) Choose the constant c as a positive real number in such a way that $||\boldsymbol{e}|| = 1$.

(b) Find a vector $\boldsymbol{e}'$, so that $\boldsymbol{e}$ und $\boldsymbol{e}'$ form an orthonormal basis for the vector space.

(c) Compute $\hat{P}_{\boldsymbol{e}}\boldsymbol{a}$, where $\hat{P}_{\boldsymbol{e}}$ is the projection onto $\boldsymbol{e}$ and $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Exercises graded according to efforts:

Problem 2.2: Complete set

Let $\hat{A}$ and $\hat{B}$ be two linear operators in a three-dimensional complex vector space. In the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, these two operators are represented as the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Are the two operators $\hat{A}$ and $\hat{B}$ hermitesch?

(b) Do the two operators $\hat{A}$ and $\hat{B}$ commute?

(c) The operator $\hat{A}$ alone does not form a “complete set”. Please explain.

(d) The operator $\hat{B}$ alone does not form a “complete set”. Please explain.

(e) Taken together, the operators $\hat{A}$ and $\hat{B}$ form a “complete set”. Find an orthonormal basis for the vector space consisting of common eigenvectors of $\hat{A}$ and $\hat{B}$.

Problem 2.3: Dirac notation

Let V be a three-dimensional complex vector space with the basis $\{|1\rangle, |2\rangle, |3\rangle\}$.

(a) A linear operator $\hat{A}$ can be represented either using the Dirac notation or as a matrix. Which matrix representation corresponds to the operator

$$\hat{A} = |2\rangle\langle 1| - |3\rangle\langle 2|?$$

(b) Compute $\hat{A}|2\rangle$.

(c) The matrix representation of the linear operator $\hat{B}$ is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the representation of this operator in the Dirac notation.

(d) Show that $\hat{B} = \hat{P}_{|1\rangle}$ is the projection onto the basis vector $|1\rangle$.

Problem 2.4: Square integrable functions

A function $F(x)$ is called "square-integrable", if the integral

$$||F||^2 = \int_{-\infty}^{\infty} dx |F(x)|^2$$

is finite. In this case $||F||$ is called the "norm" of the function F. Which of the following functions is square-integrable in one spatial dimension?

(a) $F(x) = e^{ax}$, for $a > 0$,

(b) $F(x) = e^{-ax^2+ibx}$, for $a > 0$ and real b,

(c) $F(x) = \Theta(x)\Theta(1-x)$, where $\Theta(x)$ is the "Theta Funktion", $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$.