3 Hilbert space and basic principles of quantum mechanics

Exercises graded according to correctness:

Problem 3.1: Wavefunction

Let the quantum mechanical state of a one-dimensional particle be described by the wavefunction

$$\psi(x) = \frac{1}{b} e^{-|x|/2a},$$

where $a > 0$ and b are constants.

(a) Find b so that $\psi(x)$ is normalized to unity.

(b) Find the probability density $P(x)$ for measuring the particle at the place x.

(c) Find the probability density $P(p)$ for measuring the momentum p.

Exercises graded according to efforts:

Problem 3.2: Angular momentum

Consider the operator $\hat{\boldsymbol{l}} = (\hat{l}_x, \hat{l}_y, \hat{l}_z)$, with

$$\hat{\boldsymbol{l}} = \hbar \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}},$$

where $\hat{\boldsymbol{p}} = -i\hbar\hat{\nabla}$. [It means that $\hat{l}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$, $\hat{l}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$, $\hat{l}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$.]

(a) Show that $\hat{\boldsymbol{l}}$ is hermitesch.

(b) Find the commutator $[\hat{l}_x, \hat{l}_y]$.

(c) Find the commutator $[\hat{l}_z, \hat{\boldsymbol{l}}^2]$, where $\hat{\boldsymbol{l}}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$.

(d) Find the commutator $[\hat{\Delta}, \hat{l}_z]$, where $\hat{\Delta}\psi(\boldsymbol{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(\boldsymbol{r})$.

(e) Find the commutator $[\hat{r}, \hat{l}_z]$, where $\hat{r}\psi(\boldsymbol{r}) = r\psi(\boldsymbol{r})$.

Problem 3.3: Parity operator

The parity operator $\hat{P}$ is defined as

$$\hat{P}\psi(x) = \psi(-x).$$

(a) Show that $\hat{P}$ is hermitesch.

(b) Show that $\hat{P}^2 = \hat{1}$.

(c) Find the eigenvalues of $\hat{P}$? What can be said about the eigenfunctions?

Problem 3.4: Continuous linear superposition

Consider a continuous number of functions $F_k(x)$ with delta-function normalization,

$$(F_k, F_{k'}) = \delta(k - k').$$

(a) Show that the "continuous linear superposition"

$$f(x) = \int dk \phi(k) F_k(x),$$

where $\phi(k)$ is a square-integrable function, is itself square-integrable and calculate the norm $||f||$.

(b) A specific example is the family of functions $F_k(x) = (2\pi)^{-1/2} e^{ikx}$. In this case, the continuous linear superposition is also called a "wave-packet". If $\phi(k)$ is a Gaussian,

$$\phi(k) = \frac{1}{(2\pi\sigma_k^2)^{1/4}} e^{-(k-k_0)^2/4\sigma_k^2},$$

the wave-packet is called a "Gaussian wave-packet". Calculate the function $f(x)$ for this case.