4 Principles of Quantum Mechanics

Exercises graded according to correctness:

Problem 4.1: Angular momentum

Consider the angular momentum $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$ of a quantum mechanical particle in three dimensions. In quantum mechanics, $\boldsymbol{l}$ is represented by the operator $\hat{\boldsymbol{l}} = \hbar\hat{\boldsymbol{r}}\times\hat{\boldsymbol{k}}$. In particular,

$$\hat{l}_z = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = -i\hbar\frac{\partial}{\partial\phi},$$

where (r, θ, ϕ) are spherical coordinates.

(a) The spectrum of the operator $\hat{l}_z$ consists of the values $m\hbar$, where m is an integer. What are the possible results of a measurement of the z-component of the angular momentum l_z?

(b) In general, quantum mechanics does not predict the outcome of the measurement, but only the probability for each possible result. Only for certain special states can the result of a measurement be "sharp", i.e., the result of the measurement can be predicted with 100% certainty. Is there a state $|\psi\rangle$, for which a measurement of the z-component of the angular momentum l_z is equal to $\hbar$ with a probability of 100%? If yes, give an example of the wave-function $\psi(\boldsymbol{r})$ of such a state.

(c) Is there a basis of states, in which the observable l_z is "sharp"?

(d) Is there a basis of states, in which all three components l_x, l_y and l_z are "sharp"? (If yes, then l_x, l_y and l_z are called "commensurable". If not, they are called "incommensurable".)

Exercises graded according to efforts:

Problem 4.2: One-dimensional particle

Let a "one-dimensional" particle be described by the normalized wave function

$$\psi(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}, \quad a > 0.$$

(a) What is the probability density $P(x)dx$ that the particle is found in the interval $[x, x+dx]$?

(b) Calculate the expectation value $\bar{x}$ and the spreading Δx.

(c) What is the probability density $P(p)dp$ that the result of a measurement of the momentum lies within the interval $[p, p+dp]$?

(d) Calculate the expectation value $\bar{p}$ and the spreading Δp with the help of the probability density $P(p)dp$ already computed (c).

(e) Calculate $\bar{p}$ and Δp directly, using the formulas

$$\bar{p} = \langle\psi|p|\psi\rangle, \quad \Delta p^2 = \langle\psi|(p-\bar{p})^2|\psi\rangle.$$

Problem 4.3: Conserved quantities

(a) Are the components p_x, p_y and p_z conserved for a free particle?

(b) Are the components of the position vector x, y, and z conserved for a free particle?

(c) Are the momentum components p_x, p_y and p_z conserved for a particle in a potential $V(x, y)$?

Problem 4.4: Periodic boundary conditions

Consider the Hilbert space $\mathcal{H}$ of functions $F(x)$ with periodic boundary conditions $F(x) = F(x+L)$. The scalar product on $\mathcal{H}$ is defined as

$$(F, G) = \int_0^L dx F^*(x)G(x).$$

Show that $\hat{p}_x = -i\hbar\partial/\partial x$ is Hermitian on $\mathcal{H}$.