5 Quantum mechanics in one dimension

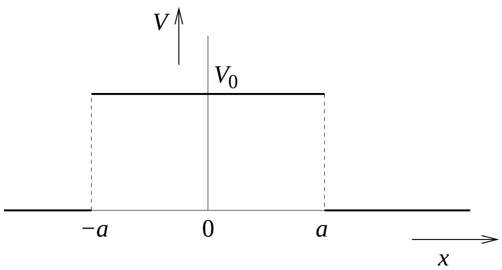
<u>*Exercises graded according to correctness:*</u>

Problem 5.1: Potential barrier

Consider a particle with mass m in a potential

$$V(x) = \begin{cases} 0 & \text{für } x < -a, \\ V_0 & \text{for } -a < x < a, \\ 0 & \text{for } x > a, \end{cases}$$

with $V_0 > 0$.



(a) Explain, why all energy-eigenvalues are non-negative $E \geq 0$.

(b) What is the general form of the energy-eigenfunctions $\psi_k(x)$ in the three regions $x < -a$, $-a < -x < a$, and $x > a$? Consider the cases $E < V_0$ and $E > V_0$ separately and use the following definitions:

$$\begin{aligned} k &= \sqrt{\tfrac{2mE}{\hbar^2}}, & K_0 &= \sqrt{\tfrac{2mV_0}{\hbar^2}}, \\ \kappa &= \sqrt{\tfrac{2m(V_0-E)}{\hbar^2}}, & K &= \sqrt{\tfrac{2m(E-V_0)}{\hbar^2}}. \end{aligned}$$

(c) What are the boundary conditions at $x = \pm a$?

(d) Explain why every energy-eigenvalue is two-fold degenerate.

<u>*Exercises graded according to efforts:*</u>

Problem 5.2: Linear Potenial

Consider a particle of mass m in a one-dimensional potential $V(x) = -cx$, where $c > 0$.

(a) Is the energy-spectrum for this particle discrete, continuous, or mixed? Justify your answer.

The energy-eigenstates of this system can be determined most easily in the momentum representation. The state $|\psi\rangle$ is then represented as

$$\psi(p) = \langle p|\psi\rangle, \quad |\psi\rangle = \int dp\psi(p)|p\rangle,$$

where $|p\rangle$ is the (δ-function normalized) momentum eigenstate with eigenvalue p.

(b) Show that the position operator $\hat{x}$ in the momentum representation has the following form:

$$\hat{x} = i\hbar\frac{\partial}{\partial p}.$$

(c) Determine the (properly normalized) energy-eigenfunctions $\psi_E(p)$ of this particle.

Problem 5.3: Tunnel effect

Consider a particle with mass m in the potential

$$V(x) = \begin{cases} 0 & \text{für } x < -a, \\ V_0 & \text{für } -a < x < a, \\ 0 & \text{für } x > a, \end{cases}$$

with $V_0 > 0$. In problem 5.1 it was shown that for each energy $E > 0$ two linearly independent energy-eigenfunctions with energy-eigenvalues $E = \hbar^2 k^2/2m$ exist. These two energy-eigenfunctions will be denoted as $\psi^+_{kR}(x)$ and $\psi^+_{kL}(x)$.

(a) Explain, why the two energy-eigenfuncions can be chosen in such a way that

$$\begin{aligned} \psi^+_{kR}(x) &= \frac{1}{\sqrt{2\pi}} \times \begin{cases} e^{ikx} + re^{-ikx} & \text{für } x < -a, \\ te^{ikx} & \text{for } x > a, \end{cases} \\ \psi^+_{kL}(x) &= \frac{1}{\sqrt{2\pi}} \times \begin{cases} te^{-ikx} & \text{for } x < -a, \\ e^{-ikx} + re^{ikx} & \text{für } x > a, \end{cases} \end{aligned}$$

(b) The factors $t(E)$ and $r(E)$ are called amplitudes of transmission and reflection. Why?

(c) Calculate the amplitudes of transmission and reflection $t(E)$ and $r(E)$. Consider the two cases $E < V_0$ and $E > V_0$ separately.

(d) Draw the coefficient $T(E) = |t(E)|^2$ as a function of E, for a potential barrier with $K_0 a = \pi$. Your drawing should clearly illustrate the behavior of $t(E)$ for $E \approx 0$, $E \approx V_0$ und $E \gg V_0$.

A very important difference between the quantum mechanical theory and the classical theory is that the quantum mechanical probability for transmission is finite for $E < V_0$: There is a small but finite probability that the particle will penetrate the potential barrier, while this is not possible in classical mechanics. This effect is called “tunnel-effect”.