6 Harmonic oscillator

Exercises graded according to correctness:

Problem 6.1: Harmonic oscillator

At time t=0, a one-dimensional harmonic oscillator of mass m and frequency ω is prepared in the state

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle - i\sqrt{2}|1\rangle + |2\rangle\right).$$

(Here, $|n\rangle$ corresponds to the standard notation in the theory of the quantum mechanical harmonic oscillator.)

(a) Calculate the expectation value of the energy.

(b) Find $|\psi(t)\rangle$.

(c) Calculate the expecation value $\overline{p}$ for the momentum p at time $t = 0$.

Hint: Express the momentum operator $\hat{p}$ with the help of the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$.

Exercises graded according to efforts:

Problem 6.2: Coherent states: Normalization and completeness

Consider the coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

for a particle in the harmonic oscillator potential, where z is a complex number and $|n\rangle$ are the energy-eigenstates, $n = 0, 1, 2, \ldots$.

(a) Calculate the scalar-product $\langle z'|z\rangle$. Are the coherent states orthonormal?

(b) Coherent states are "over-complete":

$$\int dz |z\rangle\langle z| = \pi \underline{\hat{1}}.$$

Prove this statement.

Problem 6.3: Coupled oscillators

Two identical one-dimensional oscillators with mass m and frequency ω_0 are coupled by an attractive force $F = \kappa(x_1 - x_2)$ that is proportional to the distance $x_1 - x_2$.

(a) What are the corresponding classical equations of motion and what is the classical total energy of the system?.

(b) The corresponding quantum mechanical problem is described by a wave-function $\psi(x_1, x_2)$. What is the Schrödinger equation for this problem?

(c) Transform to normal coordinates,

$$q_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad q_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

and find in this way a new Schrödinger equation. This equation can subsequently be solved exactly by separation of variables [i.e., by making the ansatz $\psi(q_1, q_2) = \psi_1(q_1)\psi_2(q_2)$]. Find the solution.

(d) What are the energy-eigenvalues for the system? What can be said about the symmetry of the eigenfunctions with respect to the interchange of x_1 and x_2?

Problem 6.4: Delta-Function potential

The transmission amplitude $t(E)$ for a one-dimensional particle with mass m, that is scattered by a δ-function potential $V(x) = -v_0\delta(x)$, is

$$t(E) = \frac{\hbar^2 k}{\hbar^2 k - imv_0}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}.$$

Show that this result coincides with the transmission amplitude $t_a(E)$ for a representation of the delta-function potential,

$$V(x) = \begin{cases} 0 & x < -a, \\ -V_0 & -a < x < a, \\ 0 & x > a, \end{cases} \qquad V_0 = \frac{v_0}{2a},$$

in the limit $a \to 0$.
Hint: for a (square) potential well one finds:

$$t_a(E) = \frac{2kK}{2kK\cos(2Ka) - i(k^2 + K^2)\sin(2Ka)}, \quad K = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}.$$