7 Quantum mechanics in three dimensions

Exercises graded according to correctness:

Problem 7.1: Rotator

Consider the three-dimensional "rotator", i.e., a system with only rotational degrees of freedom (an example is a particle that is confined to the surface of a sphere). The rotator has angular momentum $l = 1$, so that the common eigenstates $|l, m\rangle$ of $\hat{\boldsymbol{l}}^2$ and $\hat{l}_z$ can serve as basis states, where $l = 1$ and $m = 0, \pm 1$. Let the rotator be in the state

$$|\psi\rangle = \frac{1}{2}(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle).$$

(a) Calculate the expectation values of the z-component of the angular momentum l_z and of its square l_z^2.

(b) What is the probability that a measurement of the z-component of the angular momentum gives the result 0?

(c) Calculate the expectation value of the x component of the angular momentum.
Hint: Use the operators $\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y$.

Exercises graded according to efforts:

Problem 7.2: Three-dimensional harmonic oscillator

Consider the three-dimensional harmonic oscillator,

$$\hat{H} = \frac{1}{2m}\left(\hat{p}^2 + m\omega^2\hat{r}^2\right).$$

(a) Calculate the energy-eigenvalues for the three-dimensional harmonic oscillator and their degree of degeneracy.

(b) Calculate the probability distribution $P(r)$ for the distance $r = (x^2 + y^2 + z^2)^{1/2}$ to the origin $x = y = z = 0$ in the ground state.

Problem 7.3: Parity

Show that the spherical harmonics $Y_{lm}(\theta, \phi)$ are eigenfunctions of the parity operator with eigenvalue $(-1)^l$.

Problem 7.4: Rotations of a two-atomic molecule

Rotations of a two-atomic molecule can be described by a simplified model, in which the molecule is represented by two masses m_1 and m_2, which have a fixed distance a from each other.

(a) Show that in the center of mass rest frame the total energy H reduces to the kinetic energy only

$$H = \frac{\boldsymbol{l}^2}{2\mu a^2},$$

where $\boldsymbol{l}$ is the angular momentum of the two-particle system and $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass.

(b) What are the possible results of a measurement of the (quantum-mechanical) rotational energy?