8 Quantum mechanics in three dimensions (2)

Exercises graded according to correctness:

Problem 8.1: Anharmonic oscillator

Consider a one-dimensional particle with mass m in the potential

$$V(x) = \frac{1}{2}m\omega_0^2x^2 + ax^4,$$

where $a > 0$. For $a = 0$ the ground state and the first excited state are represented by the following wave functions

$$\psi_0(x) = \frac{1}{\pi^{1/4}x_0^{1/2}}e^{-(x/x_0)^2/2}$$

and

$$\psi_1(x) = \frac{x2^{1/2}}{\pi^{1/4}x_0^{3/2}}e^{-(x/x_0)^2/2}.$$

Find the lowest two energy levels of this anharmonic oscillator up to first order in a.

Excercises graded according to efforts:

Problem 8.2: Particle in a spherically symmetric potential well

Consider a particle with mass m in the central potential

$$V(r) = \begin{cases} -V_0 & \text{if } 0 \leq r < a, \\ 0 & \text{if } r \geq a, \end{cases}$$

where $V_0 > 0$. Find energy eigenstates of the time-independent Schrödinger equation

$$E\psi(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r}) + V(\boldsymbol{r})\psi(\boldsymbol{r})$$

with $E < 0$. States with $E < 0$ are bound states, the spectrum for $E < 0$ is discrete. The wave functions $\psi(\boldsymbol{r})$ can be written as

$$\psi_{ln}(\boldsymbol{r}) = Y_{lm}(\theta, \phi)\frac{f_{ln}(r)}{r},$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics. Here, m is the magnetic quantum number, l is the magnetic quantum number, and n is the principal quantum number. The energy-eigenvalues and eigenstates are labeled by n and l.

(a) Find the radial Schrödinger equation for the function $f_{ln}(r)$?

(b) Focus on the eigenstates with $m = l = 0$ (states in the so-called s-shell). What is the general form of the radial wave function $f_{ln}(r)$ in this case? Find an implicit equation, that allows to calculate the energy-eigenvalues E_{ln}. Are there bound states for arbitrary values of $V_0 > 0$?

Problem 8.3: Hydrogen atom

Calculate the expectation values $\overline{r}$ and $\overline{r^2}$ for the hydrogen atom

(a) in the ground state ($n = 1$, $l = 0$),

(b) in an excited state with $n = 2$ and $l = 1$. Does the answer depend on the magnetic quantum number m?

Problem 8.4: Rotator

Consider a "Rotator" with Hamilton-Operator

$$\hat{H} = \frac{\hat{\boldsymbol{l}}^2}{2I} + Bl_z + Cl_y,$$

where $C \ll B$.

(a) Calculate the energy-eigenvalues and their degree of degeneracy for $C = 0$.

(b) Consider the case $C \neq 0$. Use perturbation theory in order to calculate the energy-eigenvalues up to second order in C/B.

(c) Can you calculate the energy-eigenvalues for the case $C \neq 0$ exactly? If yes, compare the result of (b) with the exact solution.