9 Translations und Rotations

<u>*Exercises graded according to correctness:*</u>

Problem 9.1: Rotations

The quantum mechanical rotation operator $\hat{R}_{\eta,\boldsymbol{e}}$ rotates the state $|\psi\rangle$ by an angle η around the axis $\boldsymbol{e}$. The operator $\hat{R}_{\eta,\boldsymbol{e}}$ is related to the angular momentum operator $\hat{\boldsymbol{j}}$ via the relation

$$\hat{R}_{\eta,\boldsymbol{e}} = e^{-i\eta\boldsymbol{e}\cdot\hat{\boldsymbol{j}}/\hbar}.$$

Consider the action of this operator on a "rotator", i.e. a particle that has only rotational degrees of freedom.

(a) Show that $\hat{R}_{\eta,\boldsymbol{e}}$ does not change the magnitude of the angular momentum $\boldsymbol{j}^2$.

(b) The general normalized state of a rotator with angular momentum $j = 1$ is

$$|\psi\rangle = \sum_{m=-1}^{1} a_m |1, m\rangle,$$

where $|a_{-1}|^2 + |a_0|^2 + |a_1|^2 = 1$. Find the state $\hat{R}_{\eta,\boldsymbol{e}_z}|\psi\rangle$, which is a result of rotating the state $|\psi\rangle$ around the z-axis.

Problem 9.2: Spin 1/2 in a magnetic field

Consider a "spin 1/2", i.e., a rotator with angular momentum $j = 1/2$ in a magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$. The magnetic moment of this particle is $\boldsymbol{\mu} = \gamma\boldsymbol{s}$, where $\gamma = eg/2mc$ is the gyromagnetic ratio and $g = 2$ is the "g factor". The energy of the particle in a magnetic field is $H = -\boldsymbol{\mu}\cdot\boldsymbol{B}$. At time $t = 0$, let the spin 1/2 be in the $\hat{s}_x$-eigenstate

$$|\psi(t=0)\rangle = \begin{pmatrix} \psi_\uparrow(0) \\ \psi_\downarrow(0) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Find $|\psi(t)\rangle$ by solving the Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle.$$

(b) Find the expectation value $\overline{s_x(t)}$, $\overline{s_y(t)}$ and $\overline{s_z(t)}$.

Exercises graded according to efforts:

Problem 9.3: Translations

In an alternative approach to quantum mechanics, the momentum operator $\boldsymbol{p}$ is introduced as the generator of translations. To this end consider the operator $\hat{T}_{\boldsymbol{a}}$, that shifts the quantum mechanical state $|\psi\rangle$ by a distance $\boldsymbol{a}$,

$$|\psi\rangle \to |\psi'\rangle = \hat{T}(\boldsymbol{a})|\psi\rangle, \quad \psi'(\boldsymbol{r}, t) = \psi(\boldsymbol{r} - \boldsymbol{a}, t).$$

The momentum operator is defined by the relation

$$\hat{T}(\boldsymbol{a}) = e^{-i\boldsymbol{a}\cdot\hat{\boldsymbol{p}}/\hbar}.$$

(a) Show that this definition implies that the components $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$ of the momentum operator commute.

(b) Show that the choice of coefficient $\hbar$ in the exponent coincides with the de-Broglie hypothesis.

It is possible to consider "translations in time" as well. Let $\hat{U}(\tau)$ be the operator, that describes the time evolution of a state over a time interval τ.

$$|\psi(t + \tau)\rangle = U(\tau)|\psi(t)\rangle.$$

This operator can be used to define an operator $\hat{H}$ as

$$\hat{U}(\tau) = e^{-i\tau\hat{H}/\hbar}.$$

(c) Derive the Schrödinger equation from this definition.

(d) Show that the choice of coefficient $\hbar$ in the exponent coincides with the de-Broglie hypothesis.

Problem 9.4: Representations of the angular momentum operator

The equations

$$\hat{j}_\pm|jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle, \quad \hat{j}_z|jm\rangle = \hbar m|jm\rangle$$

imply that the components $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ of the angular momentum operator affect only the magnetic quantum number m; other quantum numbers (for example the angular momentum quantum number j or the principal quantum number n) are not affected. As a result, the components $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ can be represented by matrices for fixed j. To this end one writes a general state with angular momentum quantum number j as

$$|\psi\rangle = \sum_{m=-j}^{j} a_m|jm\rangle$$

and uses the $2j + 1$ coefficients a_m to form a vector

$$\underline{a} = \begin{pmatrix} a_j \\ a_{j-1} \\ \vdots \\ a_{-j+1} \\ a_{-j} \end{pmatrix}.$$

The components $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ of the angular momentum operator can be represented by a $2j + 1$-dimensional matrix. An example is the case $j = 1/2$, for which the operators $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ are represented by the Pauli matrices, $j_i = (\hbar/2)\sigma_i$, $i = x, y, z$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find a matrix representation for the operators $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ for

(a) $j = 0$,

(b) $j = 1$

and check that the matrix represenation fulfills the angular momentum commutation relations.