

Quantum Optimal Control for Mixed State Squeezing in Cavity Optomechanics

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The performance of key tasks in quantum technology, such as accurate state preparation, can be maximized by utilizing external controls and deriving their shape with optimal control theory. For non-pure target states, the performance measure needs to match both the angle and the length of the generalized Bloch vector. A measure based on this simple geometric picture that separates angle and length mismatch into individual terms is introduced and the ensuing optimization framework is applied to maximize squeezing of an optomechanical oscillator at finite temperature. The results herein show that shaping the cavity drives can speed up squeezed state preparation by more than two orders of magnitude. Cooperativities and pulse shapes required to this end are fully compatible with the current experimental technology.

1. Introduction

The ability to precisely control quantum systems is a prerequisite for harnessing quantum effects for quantum technology. Quantum optimal control provides a set of tools for deriving protocols to implement key tasks such as the preparation of non-classical states or the generation of entanglement.^[1] This approach can be used to determine, for example, the minimum time to carry out a given task with desired accuracy, even if the dynamics is not amenable to an analytical solution.^[2–4] Recent examples include the fast and accurate preparation of a circular state, that is, a Rydberg state with maximum projection angular momentum quantum number, for quantum sensing,^[5] and the determination of the fastest universal set of gates for quantum computing with superconducting transmon qubits.^[6]

Fast control protocols are particularly important for open quantum systems for which the interaction with the environment cannot be neglected.^[7] An obvious control strategy is to “beat” the decoherence resulting from the interaction with the environment. For quantum systems with Markovian, that is, memoryless dynamics, this is often the only option.^[8] In contrast, strongly coupled environmental modes giving rise to significant

system–environment correlations and non-Markovian dynamics are not necessarily detrimental but can also be exploited for control.^[9,10]

An alternative approach to controlling open quantum systems consists in engineering driven dissipative dynamics in such a way that the desired target state becomes the steady state of the ensuing open system evolution.^[11] This approach is particularly promising when the timescale of decoherence is comparable to or faster than that of the coherent evolution. Driven dissipative evolution is inherently robust against noise—a rather favorable feature in view of experimental implementation. In this

context, preparation of non-classical states^[12–16] and generation of non-equilibrium quantum phases^[17,18] have successfully been demonstrated with trapped atoms and ions as well as superconducting qubits. For the example of trapped ions, determining the key parameters of the driven dissipative dynamics by quantum optimal control is predicted to allow reaching the fundamental performance limits.^[19]

Another platform ideally suited for implementing driven dissipative dynamics is cavity optomechanics,^[20,21] where a mechanical resonator is coupled to an optical or microwave cavity. Optomechanical systems are promising candidates for quantum-enhanced sensing, coherent light–matter interfaces, and fundamental tests of quantum mechanics. In particular, the cavity drive can be employed to generate arbitrary quantum states of the mechanical oscillator,^[20] including strongly squeezed states. These states are useful in applications such as quantum information processing,^[22] atomic clocks^[23] or, most prominently, quantum-enhanced sensing^[24,25] where they allow to increase sensitivity of, for example, gravitational wave detectors.^[26,27] Squeezed states can be generated in various physical platforms.^[28] In cavity optomechanics, driven dissipative evolution can be used to produce substantially squeezed states.^[29,30] While the preparation of the mechanical resonator in a pure quantum state remains a major goal of cavity optomechanics, interesting non-classical states can be realized also at finite temperature.^[30–32] This is true in particular for squeezed states since there is a trade off between squeezing strength and purity.

Cavity drives for state preparation in cavity optomechanics are typically taken to have constant amplitude.^[20] On the other hand, pulsed excitation has recently been shown to allow for probing the resonator state with minimal heating.^[33] This raises the question whether explicitly time-dependent amplitudes of the cavity drives can also be used to improve state preparation protocols.

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Here, we specifically ask by how much, at a given non-zero temperature, suitable pulse shapes can speed up the preparation of the mechanical oscillator in a squeezed state.

To derive the pulse shape of the cavity drives, we employ optimal control theory (OCT) which needs to target a mixed steady state, due to finite temperature. Standard optimization functionals, defined originally for pure target states, cannot be used in this case; and alternative formulations using, for example, the Hilbert–Schmidt distance need to be employed.^[34] We give an intuitive, geometrical explanation for the failure of the standard functionals by visualizing the dynamics on the generalized Bloch sphere. This picture is also useful to elucidate the requirements an optimization functional targeting mixed states need to fulfill. Using this geometric interpretation, we furthermore refine the functional based on the Hilbert–Schmidt distance^[34] to one that seeks to match the target state’s Bloch vector angle and length separately. We employ both functionals to optimize the preparation of a mechanical resonator in a squeezed state and compare their performance.

The paper is organized as follows. The framework of quantum OCT is presented in Section 2. In Section 2.1, we briefly review Krotov’s method,^[35] our optimization algorithm of choice, in Section 2.2 we illustrate the failure of standard functionals, and in Section 2.3 we construct target functionals based on Bloch vector angle and length. Section 3 is dedicated to the application of this methodology to preparing a squeezed state at finite temperature in cavity optomechanics. We introduce the model and control problem in Section 3.1, present our results in Section 3.2, and discuss the performance of various target functionals in Section 3.3. Finally, Section 4 concludes.

2. Quantum Optimal Control Theory

Quantum control assumes that the dynamics of a quantum system can be steered, typically by a set of external driving fields $\{\mathcal{E}_k(t)\}$. How to choose the external drives in the best possible way is the subject of quantum OCT.^[1] Optimality is sought using an optimization functional:

$$J[\{\mathcal{E}_k\}, \{\hat{\rho}_i\}] = D[\{\hat{\rho}_i(T)\}] + \int_0^T dt g[\{\mathcal{E}_k(t)\}, \{\hat{\rho}_i(t)\}, t] \quad (1)$$

where $\{\hat{\rho}_i(t)\}$ is a set of forward propagated states, $D[\{\hat{\rho}_i(T)\}]$ is the figure or merit at final time T and g captures additional costs or constraints at intermediate times, for instance by restricting the field spectra or by penalizing population in certain subspaces.

A control problem is tackled by choosing i) an appropriate optimization functional J and ii) an appropriate method to minimize J . While J captures the physics of the problem, the choice of the method is relevant as well, since it often determines whether a solution can be found in practice. Gradient-based techniques hold the promise of faster convergence than gradient-free methods.^[1] However, they require the ability to determine the derivatives of the functional with respect to the states. Whether this is feasible or not depends on the choice of the functional. In the following, we focus on Krotov’s method,^[35,36] a gradient-based algorithm, but our considerations are valid for any gradient-based method requiring functional derivatives.

The general idea of any gradient-based method is to find an extremum of the total functional (1) using gradient information with respect to changes in the control fields $\{\mathcal{E}_k(t)\}$. The extremum condition together with the requirement to fulfill the dynamical equations leads to two equations of motion—one for the states $\{\hat{\rho}_i(t)\}$ and one for the so-called co-states $\{\hat{\chi}_l(t)\}$. While the former corresponds to the usual forward propagation in time starting at the initial condition $\{\hat{\rho}_i(0)\}$, the latter can be interpreted as a backward propagation in time, starting at $\{\hat{\chi}_l(T)\}$. This “initial” condition for the backward propagation contains information about the physical final time target encoded by the functional D . The optimization algorithms then tries to match both forward and backward propagated states, thus ensuring approach toward the target $\{\hat{\rho}_i(T)\}$ at final time, which in turn minimizes functional (1).

2.1. Gradient-Based Optimization with Krotov’s Method

Krotov’s method^[35,36] is a sequential optimization technique with built-in monotonic convergence. A possible choice of the functional g is^[37]

$$g[\{\mathcal{E}_k(t)\}] = \sum_k \frac{\lambda_k}{S_k(t)} (\mathcal{E}_k(t) - \mathcal{E}_k^{\text{ref}}(t))^2 \quad (2)$$

where $\mathcal{E}_k^{\text{ref}}(t)$ is a reference field (taken to be the field from the last iteration), $S_k(t) \in (0, 1]$ a shape function to smoothly switch $\mathcal{E}_k(t)$ on and off, and λ_k a parameter that controls the step size. Using Equation (2), the update equation for the field $\mathcal{E}_k(t)$ becomes^[38,39]

$$\mathcal{E}_k^{(i+1)}(t) = \mathcal{E}_k^{\text{ref}}(t) + \frac{S_k(t)}{\lambda_k} \times \Im \left\{ \sum_l \left\langle \hat{\chi}_l^{(i)}(t), \frac{\partial \mathcal{L}[\{\mathcal{E}_k\}]}{\partial \mathcal{E}_k} \Big|_{\{\mathcal{E}_k^{(i+1)}(t)\}} \hat{\rho}_l^{(i+1)}(t) \right\rangle \right\} \quad (3)$$

where $\langle \hat{A}, \hat{B} \rangle \equiv \text{Tr}\{\hat{A}^\dagger \hat{B}\}$ is the Hilbert–Schmidt overlap and $\mathcal{L}[\{\mathcal{E}_k\}]$ the field-dependent generator of the dynamics, for example, the Liouvillian of a Lindblad master equation.^[7] $\{\hat{\rho}_l^{(i+1)}(t)\}$ are forward propagated states,

$$\frac{d}{dt} \hat{\rho}_l^{(i+1)}(t) = -\frac{i}{\hbar} \mathcal{L}[\{\mathcal{E}_k^{(i+1)}\}] \hat{\rho}_l^{(i+1)}(t) \quad (4a)$$

with initial conditions

$$\hat{\rho}_l^{(i+1)}(0) = \hat{\rho}_l(0) \quad (4b)$$

whereas the co-states $\{\hat{\chi}_l^{(i)}(t)\}$ are solutions of the adjoint equation of motion

$$\frac{d}{dt} \hat{\chi}_l^{(i)}(t) = \frac{i}{\hbar} \mathcal{L}^\dagger[\{\mathcal{E}_k^{(i)}\}] \hat{\chi}_l^{(i)}(t) \quad (5a)$$

and their “initial” condition (at final time T) is determined by the target functional

$$\hat{\chi}_l^{(i)}(T) = -\nabla_{\hat{\rho}_l(T)} D \Big|_{\{\hat{\rho}_l^{(i)}(T)\}} \quad (5b)$$

The superscripts $(i + 1)$ and (i) in Equations (3)–(5) indicate the current and previous iteration in the optimization procedure, respectively. The choice of $D[\{\hat{\rho}_i(T)\}]$ enters via Equation (5b), where the derivative of functional D with respect to $\hat{\rho}_i(T)$ needs to be evaluated. For a detailed derivation of Krotov's method in the context of quantum control see ref. [38].

2.2. Failure of Overlap-Based Functionals for Mixed State Targets

State transfers, where a set of initial states $\{\hat{\rho}_i(0)\}$ must simultaneously be transferred into a set of target states $\{\hat{\rho}_i^{\text{trg}}\}$, represent a standard control problem. Solving this problem requires a reliable measure $D(\hat{\rho}_i(T), \hat{\rho}_i^{\text{trg}})$ for the state distance between forward propagated state $\hat{\rho}_i(T)$ and target $\hat{\rho}_i^{\text{trg}}$. For two pure states Ψ_1, Ψ_2 , a standard choice is^[37]

$$D_{\text{re}}(\Psi_1, \Psi_2) = 1 - \Re\{\tau\} \quad \text{or} \quad (6a)$$

$$D_{\text{sm}}(\Psi_1, \Psi_2) = 1 - |\tau|^2 \quad (6b)$$

where $\tau = \langle \Psi_1 | \Psi_2 \rangle \in \mathbb{C}$, $|\tau| \leq 1$, is the complex overlap in Hilbert space. Both functionals rely on τ to serve as a distance measure in state space, and any OCT algorithm that minimizes D_{re} or D_{sm} necessarily maximizes $\Re\{\tau\}$ or $|\tau|$, respectively. Equations (6) cannot, however, simply be generalized to non-pure states. For density matrices $\hat{\rho}_1, \hat{\rho}_2$, the overlap in Liouville space, $\tau = \langle \hat{\rho}_1, \hat{\rho}_2 \rangle \in \mathbb{R}$, defined in terms of the Hilbert–Schmidt overlap, becomes real, and minimizing $D_{\text{re}}, D_{\text{sm}}$ implies maximizing τ . Unfortunately, τ is no longer a reliable measure for closeness once $\hat{\rho}_1, \hat{\rho}_2$ are both mixed.

We illustrate the problem with the simplest example of a quantum system, a qubit. Representing the qubit state in the canonical basis, $\{|0\rangle, |1\rangle\}$, consider

$$\hat{\rho}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 - \alpha \end{pmatrix}, \quad \hat{\rho}^{\text{trg}} = \begin{pmatrix} \beta & 0 \\ 0 & 1 - \beta \end{pmatrix} \quad (7)$$

where $0 \leq \alpha, \beta \leq 1$. Both states are equivalent iff $\alpha = \beta$ and $\tau_{\text{trg}} \equiv \langle \hat{\rho}(\beta), \hat{\rho}^{\text{trg}} \rangle = \beta^2 + (1 - \beta)^2$. However, for pure states such as $\hat{\rho}(1)$ or $\hat{\rho}(0)$, $\tau = \beta$ or $\tau = 1 - \beta$. This results in $\tau > \tau_{\text{trg}}$ for $\beta \in (\frac{1}{2}, 1)$ or $\beta \in (0, \frac{1}{2})$, respectively. Thus, pure states maximize τ and thus the figure of merit (6) despite being obviously different from the target $\hat{\rho}^{\text{trg}}$. Moreover, for the completely mixed state $\hat{\rho}^{\text{trg}} = \text{diag}\{0.5, 0.5\}$, we find $\tau = \langle \hat{\rho}(\alpha), \hat{\rho}^{\text{trg}} \rangle = \frac{1}{2}$ for all α . In this case, τ is not even able to indicate differences at all.

The ill-definedness of the overlap-based functionals (6) in case of mixed target states is easily demonstrated by a toy control problem. Consider a qubit whose dynamics is determined by a purely dissipative master equation,^[7]

$$\begin{aligned} i\hbar \frac{d}{dt} \hat{\rho}(t) &= \mathcal{L}[u(t)] \hat{\rho}(t) \\ &= i\hbar u(t) \left[\hat{\sigma}_- \hat{\rho}(t) \hat{\sigma}_+ - \frac{1}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \hat{\rho}(t) \} \right] \end{aligned} \quad (8)$$

where $\hat{\sigma}_- (\hat{\sigma}_+)$ are the standard lowering (raising) operators and $u(t) \geq 0$ is a time-dependent, controllable decay rate. We choose the initial state of the qubit to be $\hat{\rho}(0) = |1\rangle\langle 1|$. Thus, we can reach any diagonal state $\hat{\rho}(T) = \alpha(T)|0\rangle\langle 0| + (1 - \alpha(T))|1\rangle\langle 1|$

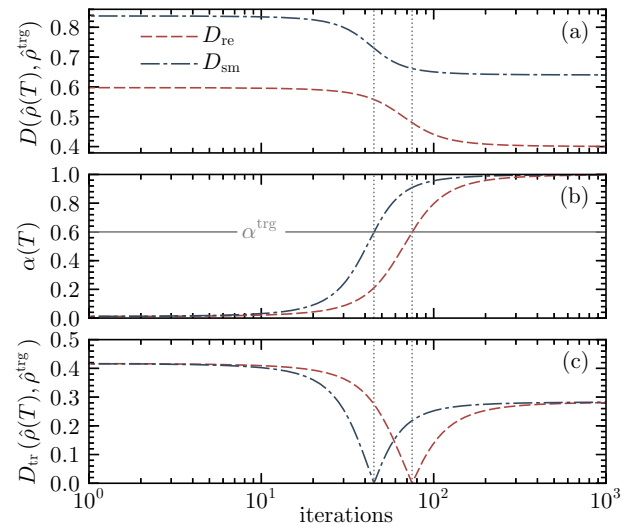


Figure 1. Optimization results for a qubit, where the dynamics is governed by Equation (8). The initial state is $\hat{\rho}(0) = |1\rangle\langle 1|$, the target state $\hat{\rho}^{\text{trg}} = \text{diag}\{0.6, 0.4\}$ and the initial field $u(t) = 0.01$ with total propagation time $T = 1$. a) Final-time functionals D_{re} and D_{sm} , cf. Equation (6), as a function of the number of iterations. b) Population $\alpha(T)$ in $|0\rangle$ at final time. The horizontal line indicates the respective population of the target state $\hat{\rho}^{\text{trg}}$. c) Trace distance D_{tr} , cf. Equation (11), between the propagated state $\hat{\rho}(T)$ and the target state $\hat{\rho}^{\text{trg}}$.

with $\alpha(T) > 0$, since $\alpha(T)$ can be controlled by appropriately choosing $u(t)$ for $t \in [0, T]$. **Figure 1** presents optimization results for a mixed state target, $\hat{\rho}^{\text{trg}} = 0.6|0\rangle\langle 0| + 0.4|1\rangle\langle 1|$, employing the functionals (6). **Figure 1a** shows the monotonic decrease of both functionals over the number of iterations, as expected for Krotov's method, while **Figure 1b** plots the corresponding final time ground state population $\alpha(T)$. The optimization starts with a fairly low ground state population, $\alpha(T) \approx 0$, due to the non-optimal, that is, too small, guess field $u(t)$. The decay rate is increased during optimization such that $\alpha(T) \approx 1$ after convergence is reached. This result maximizes the overlap since $\tau_{\text{opt}} \equiv \langle \hat{\rho}_{\text{opt}}(T), \hat{\rho}^{\text{trg}} \rangle > \langle \hat{\rho}^{\text{trg}}, \hat{\rho}^{\text{trg}} \rangle \equiv \tau_{\text{eq}}$ with $\hat{\rho}_{\text{opt}}(T) = |0\rangle\langle 0|$, and thus realizes smaller values of the functionals D_{re} and D_{sm} . However, this is not what the optimization is supposed to achieve. **Figure 1c** shows the trace distance D_{tr} (a reliable measure for the closeness of states, as we will discuss in Section 2.3) between $\hat{\rho}(T)$ and $\hat{\rho}^{\text{trg}}$ as a function of the number of iterations. A minimum of D_{tr} as the iterative algorithm proceeds, which is due to further minimization of D_{re} and D_{sm} , illustrates that the optimization misses the desired target.

This observation can be fully generalized to N -level systems with a simple geometric picture.^[40] By choosing a basis of traceless, Hermitian $N \times N$ matrices, $\{\hat{A}_i\}$, with $\langle \hat{A}_i, \hat{A}_j \rangle = \delta_{i,j}$, we can write $\hat{\rho}$ as

$$\hat{\rho} = \frac{1}{N} \hat{\mathbb{1}}_N + \mathbf{r} \cdot \hat{\mathbf{A}} \quad (9)$$

where $\mathbf{r} = (a_1, a_2, \dots)^T$ is the generalized Bloch vector, containing the expansion coefficients for matrices $\hat{\mathbf{A}} = (\hat{A}_1, \hat{A}_2, \dots)^T$.

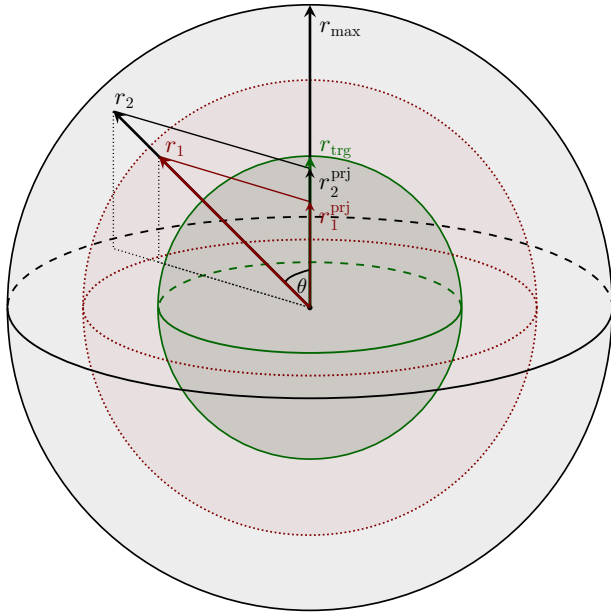


Figure 2. Bloch vectors $r_1, r_2, r_{\text{trg}}, r_{\text{max}}$ in a generalized Bloch sphere. The outer sphere indicates pure states while the inner spheres correspond to mixed states. $r_1^{\text{prj}}, r_2^{\text{prj}}$ are the projections of r_1, r_2 onto r_{trg} .

Then, the Hilbert–Schmidt overlap of two states $\hat{\rho}_1, \hat{\rho}_2$ becomes

$$\tau = \langle \hat{\rho}_1, \hat{\rho}_2 \rangle = \frac{1}{N} + r_1 \cdot r_2 = \frac{1}{N} + |r_1||r_2|\cos(\theta) \quad (10)$$

where $|r_1|, |r_2|$ are the lengths of the respective Bloch vectors and θ is the angle between them. Hence, maximization of τ means maximizing $|r_1|, |r_2|$ and minimizing θ .

Figure 2 illustrates the behavior of the overlap geometrically, comparing two states $\hat{\rho}_1, \hat{\rho}_2$ with Bloch vectors r_1, r_2 to the target state $\hat{\rho}_{\text{trg}}$ with r_{trg} . Here, we assume $r_1 \parallel r_2$ and $|r_2| > |r_1| > |r_{\text{trg}}|$. In this case, the angles θ_1, θ_2 of r_1 and r_2 with r_{trg} are identical but the purer state r_2 has the larger projection onto r_{trg} and thus yields $\tau_2 \equiv \langle \hat{\rho}_2, \hat{\rho}_{\text{trg}} \rangle > \langle \hat{\rho}_1, \hat{\rho}_{\text{trg}} \rangle \equiv \tau_1$. This contradicts the expectation that $\hat{\rho}_2$ should be further away from $\hat{\rho}_{\text{trg}}$ than $\hat{\rho}_1$ —a fact that is evidently not captured by the functionals (6). Moreover, the simple geometric picture of **Figure 2** demonstrates that the state maximizing τ is always the pure state $\hat{\rho}_{\text{max}}$ with $r_{\text{max}} \parallel r_{\text{trg}}$ and $\theta = 0$. This readily explains the optimization results of **Figure 1**.

2.3. A Bloch Vector-Based Functional for Mixed State Targets

Before inspecting specific measures for the closeness of two mixed states to replace an overlap-based functional (6), we summarize the desirable properties that a distance measure should satisfy for use in OCT. Let $S_{\mathcal{H}}$ be the space of density matrices over the Hilbert space \mathcal{H} . A function D , which quantifies the distance of two states $\hat{\rho}_1, \hat{\rho}_2 \in S_{\mathcal{H}}$, should fulfill

$$1. \forall \hat{\rho}_1, \hat{\rho}_2 \in S_{\mathcal{H}} : D(\hat{\rho}_1, \hat{\rho}_2) \in \mathbb{R}$$

$$2. D(\hat{\rho}_1, \hat{\rho}_2) = \inf_{\hat{\rho}'_1, \hat{\rho}'_2 \in S_{\mathcal{H}}} D(\hat{\rho}'_1, \hat{\rho}'_2) \Leftrightarrow \hat{\rho}_1 = \hat{\rho}_2$$

These two conditions provide the minimal framework for suitable state-to-state optimization functionals. The first property ensures that an order relation can be established. This is essential since it allows the optimization algorithm to quantify improvement in terms of a decrease of the function value. The second property guarantees that the minimal value identifies the desired state uniquely.^[41]

Evidently, the second property is not met by the overlap-based functionals D_{re} and D_{sm} . However, there exist various distance measures satisfying the two properties,^[42,43] for instance, the trace distance,^[44]

$$D_{\text{tr}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2} \|\hat{\rho}_1 - \hat{\rho}_2\|_{\text{tr}}, \quad \|\hat{\rho}\|_{\text{tr}} = \text{Tr}\{\sqrt{\hat{\rho}^\dagger \hat{\rho}}\} \quad (11)$$

the Bures distance, based on the Uhlmann fidelity,^[45,46]

$$D_{\text{Bures}}(\hat{\rho}_1, \hat{\rho}_2) = \sqrt{1 - \text{Tr}\left\{\sqrt{\sqrt{\hat{\rho}_1}\hat{\rho}_2\sqrt{\hat{\rho}_1}}\right\}} \quad (12)$$

the Hellinger distance,^[47]

$$D_{\text{Hellinger}}(\hat{\rho}_1, \hat{\rho}_2) = \sqrt{1 - \text{Tr}\left\{\sqrt{\hat{\rho}_1}\sqrt{\hat{\rho}_2}\right\}} \quad (13)$$

the Jensen–Shannon divergence,^[48]

$$D_{\text{JS}}(\hat{\rho}_1, \hat{\rho}_2) = \sqrt{E\left(\frac{\hat{\rho}_1 + \hat{\rho}_2}{2}\right) - \frac{1}{2}E(\hat{\rho}_1) - \frac{1}{2}E(\hat{\rho}_2)} \quad (14)$$

with $E(\hat{\rho}) = \text{Tr}\{\hat{\rho} \ln(\hat{\rho})\}$ the von Neumann entropy,^[49] and the Hilbert–Schmidt distance,^[50]

$$D_{\text{HS}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2} \text{Tr}\{(\hat{\rho}_1 - \hat{\rho}_2)^2\} \quad (15)$$

to name just a few. Note that D_{HS} appears in a slightly modified version to match the definition in ref. [34], and some of the other measures have been adapted to satisfy $D \in [0, 1]$. Although measures (11)–(15) fulfill the two properties, there is still one caveat left: With regard to gradient-based optimization almost all of them suffer from the fact that no closed analytical expression for their derivatives with respect to $\hat{\rho}_1$ or $\hat{\rho}_2$ exists. When resorting to numerical evaluation of the derivatives, the required spectral decomposition results in accuracy problems due to the presence of square roots, respectively logarithms, of the state. This problem becomes particularly severe in the common case that several eigenvalues of the density matrix are close to zero, rendering most of the aforementioned functionals impractical for application in gradient-based optimization.

A notable exception is the Hilbert–Schmidt distance (15), which therefore has already found use in gradient-based OCT.^[34] Motivated by the simple geometric picture of state mismatch, cf. **Figure 2**, one can also easily understand, why D_{HS} , in contrast to D_{re} or D_{sm} , is reliable. In terms of Bloch vectors, it reads

$$D_{\text{HS}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2} |r_1 - r_2|^2 \quad (16)$$

where $D_{\text{HS}} = 0$ is only attainable in case of identical Bloch vectors.

While the Hilbert–Schmidt distance mixes “angle” and “length” mismatch in a single term, one might wonder whether splitting up both contributions into two separate terms, say D_{angle} and D_{length} , allows for a more targeted optimization. We therefore define

$$D_{\text{split}}(\hat{\rho}_1, \hat{\rho}_2) = \alpha_1 D_{\text{angle}}(\hat{\rho}_1, \hat{\rho}_2) + \alpha_2 D_{\text{length}}(\hat{\rho}_1, \hat{\rho}_2) \quad (17)$$

where $\alpha_1, \alpha_2 \geq 0$ are numerical parameters that allow to weight the contributions individually, and

$$D_{\text{angle}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{\pi^2} \arccos^2 \left(\frac{d_{12}}{\sqrt{d_{11}d_{22}}} \right) \quad (18a)$$

$$D_{\text{length}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{N}{N-1} \left(\sqrt{d_{11}} - \sqrt{d_{22}} \right)^2 \quad (18b)$$

where we have used $d_{ij} \equiv \langle \hat{\rho}_i, \hat{\rho}_j \rangle - 1/N = \mathbf{r}_i \cdot \mathbf{r}_j$. In the Bloch representation, we find the simpler form

$$D_{\text{angle}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{\theta^2}{\pi^2} \quad (19a)$$

$$D_{\text{length}}(\hat{\rho}_1, \hat{\rho}_2) = \frac{N}{N-1} (|\mathbf{r}_1| - |\mathbf{r}_2|)^2 \quad (19b)$$

from which it is clear that both terms quantify angle and length mismatch individually. Measure (17) fulfills all required properties and can easily be derived with respect to $\hat{\rho}_1$ and $\hat{\rho}_2$. The derivatives read.^[51]

$$\nabla_{\hat{\rho}_1} D_{\text{angle}}(\hat{\rho}_1, \hat{\rho}_2) = -\frac{2}{\pi^2} \arccos \left(\frac{d_{12}}{\sqrt{d_{11}d_{22}}} \right) \frac{\hat{\rho}_2 - \frac{d_{12}}{d_{11}} \hat{\rho}_1}{\sqrt{d_{11}d_{22} - d_{12}^2}} \quad (20a)$$

$$\nabla_{\hat{\rho}_1} D_{\text{length}}(\hat{\rho}_1, \hat{\rho}_2) = 2 \frac{N}{N-1} \frac{\sqrt{d_{11}} - \sqrt{d_{22}}}{\sqrt{d_{11}}} \hat{\rho}_1 \quad (20b)$$

3. Application: Generation of Mixed State Squeezing

3.1. Model and Control Problem

We follow ref. [29] and consider a mode of an optical cavity and one of a mechanical resonator, coupled via radiation pressure. The optical cavity is driven by two lasers at the mechanical sidebands, $\omega_{\pm} = \omega_{\text{cav}} \pm \Omega$, where ω_{cav} and Ω are the frequencies of cavity and mechanical resonator, respectively. In the linearized regime, the Hamiltonian describing the joint system of cavity and resonator reads^[29]

$$\begin{aligned} \hat{H} = & -\hbar \hat{d}^\dagger \left(G_+ \hat{\mathbf{b}}^\dagger + G_- \hat{\mathbf{b}} \right) + \text{H.c.} \\ & - \hbar \hat{d}^\dagger \left(G_+ \hat{\mathbf{b}} e^{-2i\Omega t} + G_- \hat{\mathbf{b}}^\dagger e^{2i\Omega t} \right) + \text{H.c.} \end{aligned} \quad (21)$$

where \hat{d} ($\hat{\mathbf{b}}$) are the annihilation operators for photons (phonons). G_+ (G_-) are effective optomechanical coupling rates, given by

the optomechanical coupling constant times the amplitude of the lasers driving the blue (red) sideband of the cavity. To account for decay, we use the quantum optical master equation:^[7]

$$\begin{aligned} i\hbar \frac{d}{dt} \hat{\rho}(t) = & \mathcal{L} \hat{\rho}(t) \\ = & \left[\hat{H}, \hat{\rho}(t) \right] + i\hbar \sum_{l=1}^3 \left(\hat{L}_l \hat{\rho}(t) \hat{L}_l^\dagger - \frac{1}{2} \left\{ \hat{L}_l^\dagger \hat{L}_l, \hat{\rho}(t) \right\} \right) \end{aligned} \quad (22)$$

to describe the system’s dynamics. The Lindblad operators are given by

$$\hat{L}_1 = \sqrt{\kappa} \hat{d} \quad (23a)$$

$$\hat{L}_2 = \sqrt{\Gamma_M (n_{\text{th}} + 1)} \hat{\mathbf{b}} \quad (23b)$$

$$\hat{L}_3 = \sqrt{\Gamma_M n_{\text{th}}} \hat{\mathbf{b}}^\dagger \quad (23c)$$

with κ and Γ_M the photon and phonon decay rates, respectively, and n_{th} describing the thermal occupancy of the mechanical resonator.^[29]

Equation (22) models the driven dissipative time evolution with steady state $\hat{\rho}^{\text{th}}$. In other words, the optomechanical system will end up in $\hat{\rho}^{\text{th}}$, independent of the initial state $\hat{\rho}(0)$, if one waits sufficiently long, that is, $\hat{\rho}(0) \rightarrow \hat{\rho}^{\text{th}}$ for $t \rightarrow \infty$. The reduced steady state of the resonator alone is obtained by taking the partial trace over the cavity mode, $\hat{\rho}_{\text{res}}^{\text{th}} = \text{Tr}_{\text{cav}} \{ \hat{\rho}^{\text{th}} \}$. An appropriate choice of coupling G_- and relative strength $G_+ / G_- < 1$ results in squeezed thermal steady states of the resonator,^[29] where the squeezing strength is quantified by the expectation value $\langle \hat{X}_1^2 \rangle$ of the mechanical quadrature, $\hat{X}_1 = (\hat{\mathbf{b}} + \hat{\mathbf{b}}^\dagger) / \sqrt{2}$. It was found^[29] that larger squeezing of $\hat{\rho}_{\text{res}}^{\text{th}}$ is usually accompanied by lower purity and vice versa. The generation of strongly squeezed states comes thus at the expense of lower purity.

Note that the purity of $\hat{\rho}_{\text{res}}^{\text{th}}$, as well as that of $\hat{\rho}^{\text{th}}$ is in general determined by κ , Γ_M , and n_{th} , in addition to G_+ and G_- . In cavity optomechanics, the joint effect of these parameters is captured by the cooperativity, $\mathcal{C} = 4G_-^2 / (\kappa\Gamma_M)$. It serves as figure of merit for any optomechanical system, quantifying the exchange of photons and phonons, that is, the coupling between optical cavity and mechanical resonator.^[20,52]

If the laser drives operate continuously, the time T it takes to reach $\hat{\rho}^{\text{th}}$ with sufficient accuracy is essentially determined by the cooperativity \mathcal{C} and the optomechanical coupling rates G_+ , G_- . Assuming cavity and resonator to be initially in thermal equilibrium, we may ask whether it is possible to accelerate the approach of the steady state by suitably shaping the drives. To this end, we consider time-dependent driving strengths of the blue and red sideband tones. This results in time-dependent effective coupling rates $G_-(t)$ and $G_+(t)$. We will use optimal control theory as outlined in Section 2 to determine shapes of $G_-(t)$ and $G_+(t)$ that allow for a faster approach to the steady state compared to the constant drives of ref. [29].

3.2. Speeding up the Approach of the Steady State

The assumption of thermal equilibrium initially corresponds, for the cavity, to the ground state, $^{[53]} \hat{\rho}_{\text{cav}}(0) = |0\rangle\langle 0|$, whereas the initial state of the resonator is characterized by the thermal occupancy n_{th} , for which we choose $n_{\text{th}} = 2$.^[54] Cavity and resonator decay rate are taken from the experiment reported in ref. [30], that is, $\kappa/2\pi = 450$ kHz, $\Gamma_M/2\pi = 3$ Hz. The target state for the optimization is given by the state obtained with the time-continuous protocol of ref. [29] after 15 ms which is virtually identical to the steady state. For the given parameters, the squeezing strength in the steady state, $\langle \hat{X}_1^2 \rangle_{\text{ZPF}} / \langle \hat{X}_1^2 \rangle$, amounts to approximately 5.7 dB which is beyond the 3 dB limit. Constant values for G_- and G_+ , cf. ref. [29], are taken as a guess pulse for starting the iterative optimization. In detail, we choose G_+ and G_- such that $\mathcal{C} = 100$ and $G_+/G_- = 0.7$, since this balances well squeezing and mixedness of the associated steady state. In the calculations, $N_{\text{res}} = 40$ and $N_{\text{cav}} = 4$ levels for resonator and cavity mode turn out to be sufficient to prevent reflection due to the finite Hilbert space size. Fast oscillating terms in Equation (21) have been neglected, which is, given our choice of \mathcal{C} , in accordance with ref. [29].

Figure 3 compares the dynamics of the time-continuous protocol of ref. [29] (blue solid lines) to those induced by optimized drives, using several target functionals and a total time of 1 ms. The joint state purity and resonator squeezing are analyzed in Figure 3a,d, respectively. Moreover, Figure 3b shows the difference $(G_-^2(t) - G_+^2(t))^{1/2}$, which determines an effective cooling rate into the squeezed state while Figure 3c shows the ratio $G_+(t)/G_-(t)$, an effective rate determining the squeezing strength of the final steady state, cf. ref. [29]. Pulses optimized using D_{HS} or D_{split} result in an acceleration of the thermalization process, cf. the blue solid versus purple double-dashed, brown dotted, and green dashed-double dotted lines in Figure 3a. These lines all converge to the proper joint state purity. Similarly, the resonator squeezing reaches the desired value for the corresponding curves in Figure 3d and does so significantly faster for all optimized pulses.

Inspection of Figure 3b,c allows us to unravel the control strategy. It consists, independently of the target functional, in an increase of the effective cooling rate $(G_-^2(t) - G_+^2(t))^{1/2}$ in order to speed up the cooling into the (squeezed) steady state. In general, ramping up G_- and G_+ will always accelerate the coherent part of the dynamics, since the norm of the Hamiltonian (21) determines the timescale of the system's coherent dynamics. However, ramping up the coupling also changes the steady state of the driven dissipative dynamics. Thus, the increase of $(G_-^2(t) - G_+^2(t))^{1/2}$, which is in our case in fact achieved by increasing both $G_-(t)$ and $G_+(t)$, needs to be balanced by a modulation of $G_+(t)/G_-(t)$ to ensure steering the system toward the correct target state. Interestingly, the optimizations with both D_{HS} (purple double-dashed lines) and D_{split} (green dashed-double dotted and brown dotted lines) find almost identical control fields. This is not guaranteed due to non-uniqueness of the control solution in most cases and indicates that we explore comparable optimization landscapes^[55,56] despite the different functionals.

Note that the optimized control fields of Figure 3b,c only require a slow modulation of the drive amplitudes while keeping their frequencies constant. This makes them experimentally feasible with existing technology—such slow modulations can easily

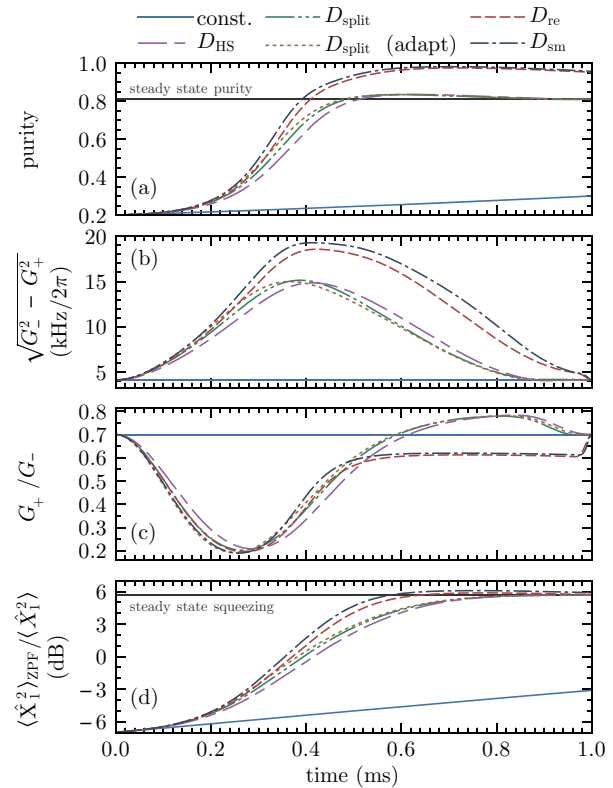


Figure 3. Comparison of the time evolution obtained with time constant (blue solid lines) and optimized drives, using the target functionals indicated in the legend: a) joint state purity, b,c) optimized effective cooling rate $(G_-^2(t) - G_+^2(t))^{1/2}$ and squeezing rate $G_+(t)/G_-(t)$, respectively, d) squeezing strength $\langle \hat{X}_1^2 \rangle$ compared to the zero-point fluctuations $\langle \hat{X}_1^2 \rangle_{\text{ZPF}}$ in dB, that is, $10 \cdot \log_{10}\{\langle \hat{X}_1^2 \rangle_{\text{ZPF}} / \langle \hat{X}_1^2 \rangle\}$. “ D_{split} (adapt)” represents an optimization using an adaptive choice for the weights of angle and length, cf. Equation (24).

be realized by arbitrary waveform generators allowing for amplitude modulations on timescales down to subnanoseconds^[57] or even significantly more complex pulse shapes, see for example, ref. [58] for one example.

Another concern that often arises in the context of experimental feasibility of optimal control protocols is robustness with respect to noise in the controls. We therefore examine whether our optimized drives are robust with respect to amplitude noise. To this end, we apply 0.2, 0.5, and 1.0% constant noise to the optimized field shapes^[59] by rescaling the field amplitudes accordingly. We obtain for the final trace distance D_{tr} with respect to the targeted squeezed steady state an average of 3.7×10^{-4} , 9.2×10^{-4} , and 1.8×10^{-3} , respectively. This needs to be compared to $D_{\text{tr}} = 4.8 \times 10^{-6}$ for the original optimized fields and to $D_{\text{tr}} = 2.6 \times 10^{-2}$ which one would obtain under the evolution with constant drives up to that point in time. The increase of the absolute error, of the order of $10^{-4} - 10^{-3}$, is not surprising given the order of the noise which is $10^{-3} - 10^{-2}$.

Figure 4a provides a closer look at the asymptotic squeezing dynamics of Figure 3d, showing that only pulses optimized with D_{HS} and D_{split} reach the correct squeezing at final time, cf. the purple double-dashed and brown dotted lines. In contrast, the

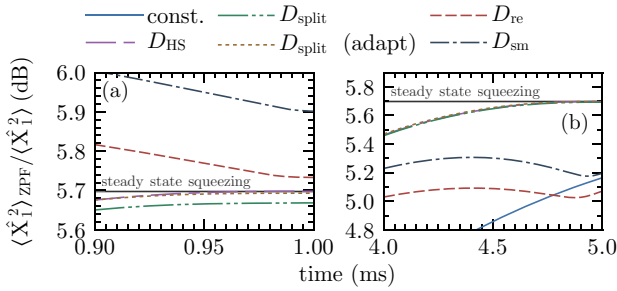


Figure 4. a) The relevant part of the squeezing dynamics as shown in Figure 3d. b) A similar dynamics as in (a) but for an optimization with final time 5 ms (optimized fields and dynamics not shown).

fields optimized with D_{re} and D_{sm} (red dashed and dark blue dot-dashed lines in Figure 3) fail to steer the system toward a state with the correct purity. Instead, they act in order to increase the purity at final time T as much as possible, failing to reach, however, completely pure states which are not attainable due to the finite temperature ($n_{th} > 0$). Figures 3 and 4a thus illustrate once more that the target functionals D_{re} and D_{sm} should not be used for non-pure target states.

Interestingly, the dynamics shown in Figure 4a all result in a comparable squeezing with the final values $\langle \hat{X}_1^2 \rangle_{ZPF} / \langle \hat{X}_1^2 \rangle$ obtained with D_{re} and D_{sm} even beyond the intended steady state squeezing of roughly 5.7 dB, cf. the red dashed and dark blue dot-dashed lines in Figure 4a. This indicates a larger squeezing to be possible than the one set by the steady state with its corresponding tradeoff between squeezing strength and purity. The apparently “good” optimization results with respect to the final-state squeezing obtained with D_{re} and D_{sm} in Figure 4a can be explained by the fact that squeezing of any state is mainly determined by its direction on the generalized Bloch sphere. Here, the optimization benefits from the fact that D_{re} and D_{sm} try to match the final state directions. However, this is not always the case, as illustrated in Figure 4b showing the squeezing dynamics for a similar optimization as in Figure 3 but with a final time of 5 ms. Note that for the constant protocol of ref. [29], the optimal relation G_+/G_- of driving strengths which maximizes the squeezing strength can be estimated by n_{th} and the cooperativity C . However, this estimation is no longer easily possible for shaped driving fields since they give rise to time-dependent cooperativities $C(t)$.

If the state preparation errors obtained with the optimized fields after 1 ms are not yet sufficient, it should be possible to continue approaching the steady state using the original protocol^[29] of constant drives. This is examined in Figure 5 which shows the evolution of the trace distance D_{tr} , cf. Equation (11), under constant drives and optimized fields from Figure 3b,c, switched back to constant drives at $T = 1$ ms. D_{tr} continues to decrease for times larger than the switching time, that is, the final time used in the optimization. A monotonous decrease of D_{tr} across the switching time, as observed in Figure 5, is expected for the fields optimized with D_{split} and D_{HS} . It does not need to be the case, however, for the fields optimized with D_{re} or D_{sm} . Here, the state at the switching time, although already closer to the target state than with constant driving, is still comparatively far from the steady state. Nevertheless, upon subsequent propagation with

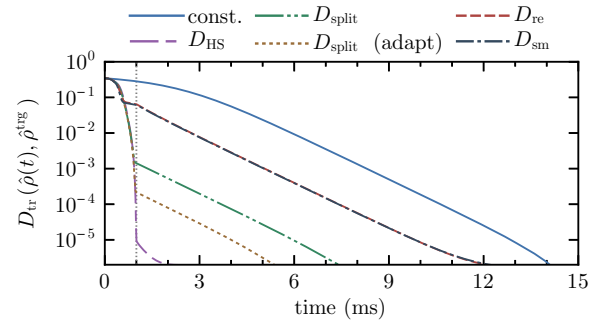


Figure 5. Dynamics of the trace distance D_{tr} , cf. Equation (11), under the guess and optimized fields of Figure 3. Beyond $T = 1$ ms (indicated by vertical line) all optimized fields are extended by the constant fields of the original, time-independent protocol.

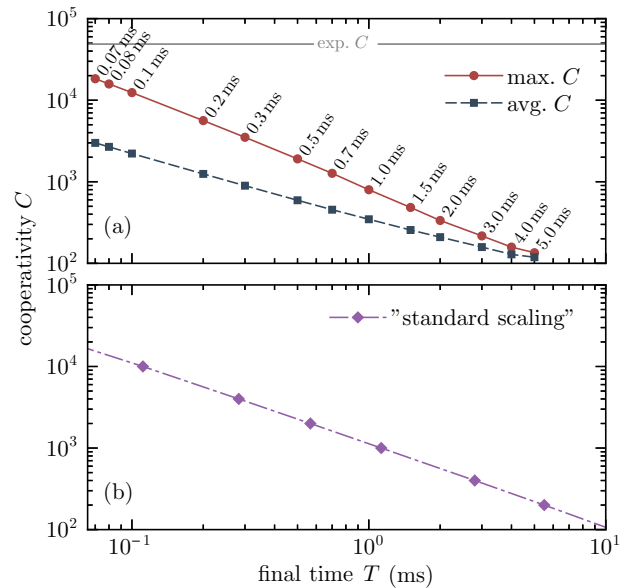


Figure 6. a) Peak and average cooperativity C , calculated from the optimized fields $G_-(t)$, required to achieve a state preparation error of at most $D_{tr} < 10^{-4}$, as function of the total optimization time T (employing D_{HS} for all optimizations). The horizontal line indicates the static cooperativity used in the experiment reported in ref. [30]. b) Minimal time against required cooperativity to reach a steady state with $D_{tr} < 10^{-4}$ for the original, time-constant protocol of ref. [29]. Note that each point in (b) corresponds to a different steady state while all points in (a) correspond to the same steady state.

constant drives, D_{tr} is further improved in all cases. Figure 5 thus provides another illustration of the speed up in preparing the squeezed steady state.

Finally, Figure 6 answers the question by how much the approach of the steady state can be accelerated. The price for speed-up is cooperativity, or, in other words, laser intensity, as illustrated by Figure 6a. It shows the peak and average cooperativity C , determined by the optimized field $G_-(t)$, as a function of the total optimization time T . Given an experimental bound on the cooperativity C , one can thus easily determine the required time T to reach the target state. Taking the experimental value of the cooperativity reported in ref. [30], we find a speedup of at least two orders of magnitude, see Figure 6a, compared to

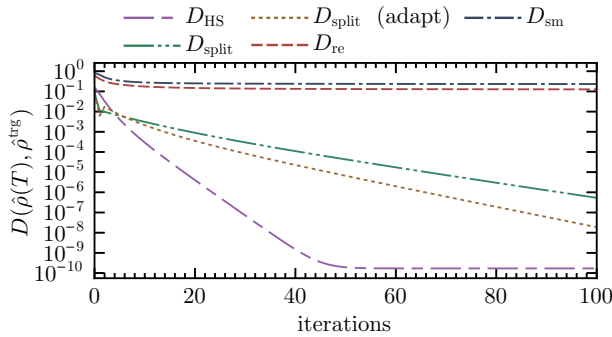


Figure 7. Convergence behavior of optimization algorithm for the target functionals used in Figure 3. The weighting of the two terms in D_{split} was chosen as $\alpha_1 = \alpha_2 = 1/2$, cf. Equation (17), for the green dashed double-dotted line, while it was adapted after each iteration for the brown dotted line (see text).

the original protocol employing constant drives.^[29] Conversely, fixing a certain duration T determines the required cooperativity or laser power. Durations as short as $T = 0.07$ ms are feasible, while the state preparation errors are still sufficiently small with $D_{\text{tr}} < 10^{-4}$ for all points in Figure 6a. Moreover, the optimized pulse shapes corresponding to the data from Figure 6a all look quite similar to the ones presented in Figure 3b,c.

Both peak and average cooperativity of the optimized field increase with decreasing duration, as one would expect for reaching the same target in less time. We observe an almost perfect power law dependence of the cooperativity \mathcal{C} as a function of the duration T in Figure 6a. This power law should be compared to the intrinsic scaling of the system due to its non-linearity which is shown in Figure 6b. Note that each point in Figure 6b corresponds to a different steady state while all points in Figure 6a correspond to the same steady state, namely the one used as benchmark in Figure 3 after 15 ms. The similar scaling observed in Figure 6a,b thus indicates the system non-linearity to be the defining feature even in the case of time-dependent and optimized drives.

Note that the short fields with cooperativities $\mathcal{C} \approx 10^4$ approach the regime where the rotating wave approximation starts to break down.^[29] Hence, we chose not to examine shorter, respectively stronger fields in Figure 6.

3.3. Performance of Optimization Functionals

The convergence behavior of the optimization algorithm for the various target functionals used in Figure 3 is inspected in **Figure 7**. The functional value of D_{re} and D_{sm} rapidly approaches a plateau which indicates that the optimization got stuck and no improvement with respect to the guess pulses could be realized. In contrast, optimizations with D_{HS} (purple double-dashed line) and D_{split} (green dashed-double dotted and brown dotted lines) show an improvement of several orders of magnitude. For the optimizations with D_{split} , we have used two different variants. For the green dashed double-dotted line constant, equivalent weights α_1, α_2 , cf. Equation (17), have been used, while for the brown dotted line we have employed an automated update scheme for the weights after each iteration. In the latter case, we have adjusted

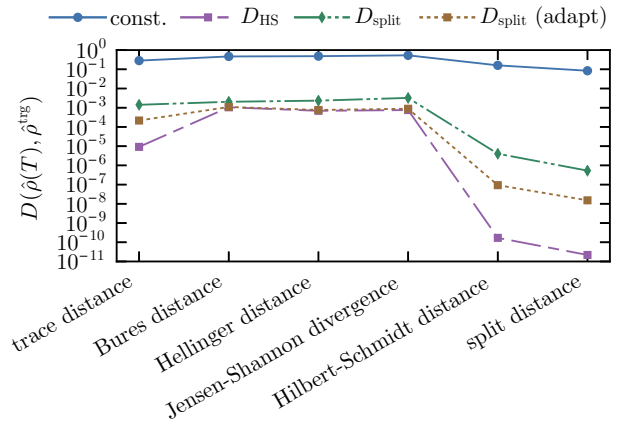


Figure 8. Comparison between changes in various distance measures under optimization with D_{HS} and D_{split} , cf. Figure 3.

the weights for the next iteration $i + 1$:

$$\alpha_1^{(i+1)} = \frac{D_{\text{angle}}^{(i)}}{D_{\text{angle}}^{(i)} + D_{\text{radius}}^{(i)}}, \quad \alpha_2^{(i+1)} = \frac{D_{\text{radius}}^{(i)}}{D_{\text{angle}}^{(i)} + D_{\text{radius}}^{(i)}} \quad (24)$$

using values of the current iteration i . This effectively causes the dominating term to become preferentially minimized within the next iteration. Although breaking strict monotonic convergence of Krotov's method over multiple iterations, due to optimizing a different functional in each iteration, it yields better convergence in our example. The plateau of D_{HS} at $\approx 10^{-10}$, starting at iteration ≈ 50 , is not of physical origin but caused by the propagation accuracy; choosing a finer time discretization would probably allow the optimization to reach even smaller values.

Note that the scaling parameters λ_k , cf. Equation (2), have been individually chosen for the different functionals in all optimizations shown in Figure 7.^[60] The necessity of different scalings is readily explained by the co-states $\hat{\chi}_l(T)$, since their norm influences the magnitude of the field updates, via Equation (5b). Due to different norms for different functionals, the optimization parameters λ_k must usually be adjusted when switching functionals if one wants to maintain field updates of similar magnitude.

In the same context, one might naively conjecture from Figure 7 that, because D_{HS} yields smaller functional values than D_{split} , D_{HS} yields better optimization results. However, such a statement would in general be wrong. As discussed above in Section 2.3, the accuracy with which the target state is reached is not uniquely assessed by a single measure. A small value of D_{HS} does not necessarily imply a similarly good value for any other distance measure. **Figure 8** therefore displays the value of several reliable distance measures for the final state obtained with the fields optimized using D_{HS} and D_{split} and compares it with the non-optimized protocol, that is, constant driving (blue line). We indeed observe that the optimization with D_{HS} (purple double-dashed line) yields the smallest state-preparation errors also for all other distance measures in Figure 8.

This does not need to hold in general, however, since the *absolute* value of any distance measure D is not to be confused with *relative* physical closeness of two states. While the measures D considered in Figure 8 are all known to be reliable, they

assess state mismatches differently for $D > 0$. For instance, if $D(\hat{\rho}_2, \hat{\rho}^{\text{trg}}) > D(\hat{\rho}_1, \hat{\rho}^{\text{trg}}) > 0$ for two states $\hat{\rho}_1, \hat{\rho}_2$, a desired target state $\hat{\rho}^{\text{trg}}$, and a measure D does not imply the same to be true for another measure \tilde{D} . In other words, two reliable measures can still disagree on which of two states is closer to the target even though they both correctly assess when a state becomes identical to the target.

For the presented problem, the performance of the Hilbert–Schmidt distance D_{HS} compared to the split-functional D_{split} is slightly better, cf. Figures 7 and 8. Nevertheless, D_{split} contains information about angle and length mismatch of the Bloch vectors individually and thus provides more insight into the dominating source of mismatch which D_{HS} cannot provide. While this information was not of relevance here it could certainly be of interest for other optimization problems.

4. Conclusions

We have studied how to speed up evolution toward a squeezed steady state in a driven optomechanical system, consisting of cavity and mechanical resonator coupled via radiation pressure. To this end, we have replaced the constant drives of the original protocol^[29] by time-dependent pulses and derived the corresponding pulse shapes using quantum optimal control theory. To the best of our knowledge, our work is the first to apply quantum optimal control to cavity optomechanics. Further potential of quantum optimal control for this popular experimental platform is highlighted by a recent proposal suggesting to couple the cavity additionally to a two-level system in order to drive the mechanical oscillator into a Fock state.^[61]

Our control solutions for accelerating the approach of a squeezed steady state consist in increasing the effective optomechanical coupling at intermediate times. At final time, the value of the constant coupling is resumed, ensuring approach of the proper steady state. We find the cooperativity corresponding to the increased optomechanical coupling due to the optimized fields to grow polynomially with decreasing protocol duration, for both average and peak value. Limiting the maximum cooperativity to that of the experiment reported in ref. [30], a speed up of more than two orders of magnitude is possible, compared to the protocol using constant drives. The required pulse shapes correspond to simple modulations and are feasible with current technology using, for example, arbitrary wave form generators. In view of using the squeezed state, for example in quantum sensing, such a speed up will be important to minimize the detrimental influence of decoherence.

Since the steady state balances quantum mechanical purity and resonator squeezing, the control problem targets a non-pure state, and care must be taken when defining the target functional. In particular, functionals based on state overlaps fail when both states—the true state and the target state—are mixed. A possible remedy consists in replacing the overlap by a (modified) distance measure.^[34] We have visualized the failure of overlap based functionals by examining the state vectors on the Bloch sphere: While the overlap only seeks to match the angle, a reliable figure of merit needs to match both angle and length of the vectors. This geometric picture provides the intuition for defining an alternative target functional, based on matching angle and length

of the Bloch vectors separately. We have successfully employed this target functional as an alternative to a functional based on the Hilbert–Schmidt distance,^[34] obtaining fairly similar solutions to the control problem at hand. Moreover, we observe that optimization with both functionals not only leads to a minimization of the respective distance measure that is being employed but also to a reduction of any other distance measure that can be used to assess the state preparation error.

Our results of accelerated state preparation are relevant when exploiting squeezed states, for example in quantum sensing. Moreover, our Bloch vector based target functional should be useful, in general, to estimate quantum speed limits.^[62] While the mismatch in Bloch vector angles quantifies rotation (i.e., unitary) errors, that in Bloch vector length estimates dissipative errors. If, for a given system, one can find an expression for the evolution speed of Bloch vector angle and length, this would allow to determine separate quantum speed limits for the unitary and dissipative parts of a system's evolution. One could thus decide which of the two sets the overall speed limit.

Our results also give rise to an interesting further question in the context of squeezed state preparation. Incidentally, we have found fields that, while not resulting in the correct steady state, produce larger squeezing than expected for the steady state, with higher purity. This suggests to directly maximize the squeezing at final time, irrespective of the state at that time, instead of targeting a specific squeezed state as we have done here. Such an optimization is possible by taking the expectation value of the relevant quadrature as target functional. It would allow to examine the conditions for avoiding the trade-off between purity and squeezing to which the steady state is subject to^[29] and, more generally, determine the ultimate limit of quantum mechanical squeezing.

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

cavity optomechanics, mixed state squeezing, quantum optimal control

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