Advanced statistical mechanics

(WS 12/13, FU Berlin)

Problem sheet 1

Due date: October 31, 2012

Brief information

- The problem sheets will be distributed always on Tuesdays.
- Each problem sheet will allow for earning 16-20 points.
- The solutions can be handed in by groups of two students.
- You are permitted to work together with more students, but then indicate all their names on your completed homework.
- The solution is to be handed in at latest in the tutorial on the due date. If solutions are repeatedly (more than once) handed in late only half of the achieved point will be awarded.
- It is necessary to have achieved at least 60 percent of all points in order to be allowed to participate in the exam.
- The exam will determine the final grade.
- Tutorials are on Wednesday at 2pm and 4pm in room E1.

Problems

Problem 1: Exact and closed differentials (4+3)

a) Consider the following differentials on $\mathbb{R}^2$:

$$\delta A_1 \coloneqq (2y^2 - 3x)\,\mathrm{d}x - 4xy\,\mathrm{d}y. \tag{1}$$

$$\delta A_2 \coloneqq (y - x^2)\,\mathrm{d}x + (x + y^2)\,\mathrm{d}y \tag{2}$$

$$\delta A_3 \coloneqq (y - x^2)\,\mathrm{d}x - (x + y^2)\,\mathrm{d}y \tag{3}$$

$$\delta A_4 \coloneqq \frac{x\,\mathrm{d}y}{x^2 + y^2} - \frac{y\,\mathrm{d}x}{x^2 + y^2} \tag{4}$$

Are these differentials closed? Are they exact? Prove it and find a potential if possible!

b) Consider the two differentials on $\mathbb{R}^2$:

$$\delta A_5 \coloneqq (2xy + x^2)\,\mathrm{d}x + x^2\,\mathrm{d}y \tag{5}$$

$$\delta A_6 \coloneqq y(x - 2y)\,\mathrm{d}x - x^2\,\mathrm{d}y. \tag{6}$$

For both differentials, find the change in A between a fixed point, $(a, b) \in \mathbb{R}^2$ and the point (x, y). Compute the change in two different ways:

(i) Integrate along the path $(a, b) \to (x, b) \to (x, y)$, and

(ii) integrate along the path $(a, b) \to (a, y) \to (x, y)$.

Discuss the meaning of your results.

Problem 2: Thermodynamic cycle (1+2+2+3)

Electromagnetic radiation in an evacuated vessel of volume V at equilibrium with the walls at temperature T (black body radiation) behaves like a gas of photons having internal energy $U = aVT^4$ and pressure $p = aT^4/3$, where a is the radiation constant.

a) Determine the curve $V \mapsto p(V)$ in the p-V plane for an isothermal process.

b) Show that for an adiabatic processes the curve $V \mapsto p(V)$ can be written as

$$p(V) = cV^{-\kappa}, \tag{7}$$

where c is some constant and κ the adiabatic coefficient. Determine the value of κ.
Hint: The work differential is $\delta A = p\,dV$.

c) Plot or sketch the closed curve of a Carnot cycle using blackbody radiation. How are the four subprocesses called? Which of the two adiabatic processes has the larger constant c?

d) Derive *explicitly* the efficiency of a Carnot engine using blackbody radiation as its working substance.

Problem 3: Clausius' statement (3) says that "heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time." Prove this statement by showing that such a process could be used to build a perpetual motion of the second kind.
Hint: It can be helpful to use a diagram. Nevertheless give all equations necessary for a complete proof.