Advanced statistical mechanics

(WS 12/13, FU Berlin)

Problem sheet 2

Due date: November 7, 2012

Problem 4: Efficiency of heat engines (1+2+2+2+1)

Carnot engines work quasi statically, i.e., infinity slowly and thus have no output power. For a real heat engine one is more interested in maximizing the output power than in maximizing the efficiency. A more realistic model of a heat engine, which takes into account the time necessary to transfer the heat from/to the reservoirs to/from the working medium, is the following:

The hot and cold reservoirs have temperatures T'_+ and T'_- respectively. The engine performs a Carnot cycle. During the isothermal expansion (compression) the working medium of the machine is held at $T_+ < T'_+$ ($T_- > T'_-$) and the heat Q_1 (Q_3) is transfered from (to) the reservoir in the time interval t_1 (t_3), i.e.,

$$Q_1 = k(T'_+ - T_+)t_1 \tag{1}$$
$$Q_3 = k(T'_- - T_-)t_3, \tag{2}$$

where k is the thermal conductivity. We want to find the T_+ and T_- for which the machine gives the optimal output power.

The time for the adiabatic processes is neglected. Even though the problem can be solved in the general case you can assume here for simplicity that $t_3 = t_1$.

a) Does the power depend on the value of t_1?

b) Combine the first and second Law to write T_+ as a function of T_-, T'_+, and T'_-.
Hint: It turns out that $T_+ = T'_+/(2 - T'_-/T_-)$.

c) Use this expression to write the power P as a function of T_- and T'_+, T'_-.

d) Determine the optimal values of T_- and T_+ as a function of only T'_+ and T'_-.
Hint: It turns out that $T_- = (T'_- + \sqrt{T'_- T'_+})/2$.

e) *Bonus question:* Calculate the efficiency of the heat engine as a function of T'_+ and T'_-.

Problem 5: Legendre transform I (1+1+2)

The Legendre transform $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\tilde{f}(\xi) := \sup_{x \in \mathbb{R}^n} \{\xi \cdot x - f(x)\}. \tag{3}$$

a) Prove that the Legendre transform $\tilde{f}$ is always convex.

Now let $f : \mathbb{R} \to \mathbb{R}$ be a smooth strictly convex function.

b) Prove that $\tilde{f}$ satisfies the functional equation

$$\tilde{f}(f'(x)) = x f'(x) - f(x). \tag{4}$$

Hint: Calculate the maximum in Eq. (3).

Importantly, Eq. (4) implies that the Legendre transform of a smooth strictly convex function f is $\tilde{f} = f^\star$, where

$$f^\star(\xi) := \left[f'\right]^{-1}(\xi)\,\xi - f\big(\left[f'\right]^{-1}(\xi)\big), \tag{5}$$

where $[f']^{-1}$ denotes the inverse function of f'. Often Eq. (5) is take as a definition for the Legendre transform.

c) Prove that

$$f = f^{\star\star}, \tag{6}$$

i.e., on smooth strictly convex functions the Legendre transform is its own inverse.
Hint: Show first that $[f^\star]' = [f']^{-1}$.

Problem 6: Legendre transform II (4)

Consider the following functions:

- $f_1(x) = \mathrm{e}^{\alpha x}$ for some constant $\alpha \in \mathbb{R}$ and
- $f_2(x) = c\,x$ for some constant $c \in \mathbb{R}$.

Find both, $\tilde{f}_j$ and $f_j^\star$ for $j = 1, 2$ and for *all* possible constants $\alpha \in \mathbb{R}$ and $c \in \mathbb{R}$. Do they coincide? If not then explain why.

Problem 7: Thermodynamic potentials (3+1+1)

Consider a magnetic system for which the internal energy differential is $dU = TdS + MdB$, where M is the magnetization and B is an externally applied magnetic field. The internal energy is written as a function of the independent quantities S and B.

Define A to be the thermodynamic potential that is the negative of the Legendre transform of U with respect to S, i.e.,

$$A(\xi, B) := -\sup_{S\in\mathbb{R}}\{\xi S - U(S, B)\}. \tag{7}$$

We assume that $U(S, B)$ is a smooth strictly convex function.

a) Find $\frac{\partial U(S,B)}{\partial S}$, and hence, (using the relation Eq. (4),) show that

$$dA = MdB - SdT \qquad \text{(as a function of } T \text{ and } B\text{)}. \tag{8}$$

In particular, write down all variable transformations explicitly!

b) Use dA to find a relationship between the rate of change of entropy (w.r.t T and at constant B) and the rate of change of M (w.r.t B and at constant T).

c) Assuming reversible operations, under what conditions is A constant?

Brief information

- The problem sheets will be distributed always on Tuesdays.
- Each problem sheet will allow for earning 16-20 points.
- The solutions can be handed in by groups of two students.
- You are permitted to work together with more students, but then indicate all their names on your completed homework.
- The solution is to be handed in at latest in the tutorial on the due date. If solutions are repeatedly (more than once) handed in late only half of the achieved point will be awarded.
- It is necessary to have achieved at least 60 percent of all points in order to be allowed to participate in the exam.
- The exam will determine the final grade.