Advanced statistical mechanics
(WS 12/13, FU Berlin)

Problem sheet 5

Due date: November 28, 2012

Problem 15: Change of variables (2+1)

We consider a probability measure $\mathbb{P}$ on the sample space Γ, two random variables

$$X : \Gamma \to G \subset \mathbb{R}^n \tag{1}$$
$$Y : \Gamma \to F \subset \mathbb{R}^n, \tag{2}$$

and a diffeomorphism (smooth, invertible with smooth inverse) $g : G \to F$ with

$$Y = g(X) := g \circ X. \tag{3}$$

a) We assume that $\mathbb{P}_Y$ has a probability density function ρ_Y, i.e., for all events $B \subset F$

$$\mathbb{P}_Y[B] = \int_B \rho_Y(y)\ \mathrm{d}y. \tag{4}$$

Prove that the probability density function ρ_X of $\mathbb{P}_X$ is given by

$$\rho_X(x) = \rho_Y(g(x))\, |\det(\mathrm{D}_x g)|. \tag{5}$$

Hint: Use integration by substitution.

For simplicity, you can now assume, that X and Y are one dimensional random variables, i.e., $n = 1$.

b) Now we assume that ρ_X is given and the goal is to find ρ_Y. Show that the probability density function ρ_Y of $\mathbb{P}_Y$ is given by $\rho_Y(y) = \rho_X(g^{-1}(y))\ \left|[g^{-1}]'(y)\right|$, i.e., with the notation $y = g(x)$,

$$\rho_Y(y) = \rho_X(x) \left|\frac{\partial y}{\partial x}\right|. \tag{6}$$

Problem 16: Basic probability theory – a simple example (1+1+2+1+1)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 + x & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

a) Find a constant Z such that $\rho := f/Z$ is a probability density function.

b) Denote the probability measure given by ρ by μ. Find a function F such that for every interval $[a, b] \subset \mathbb{R}$

$$\mu([a, b]) = F(b) - F(a). \tag{8}$$

This function is called *distribution* of μ.

Consider the random variables $X, Y : \mathbb{R} \to \mathbb{R}$ given by $X(x) = x$ and $Y(x) = 2x + x^2$.

c) Calculate their means and variances.

d) What is the probability density function ρ_X and the distribution F_X of the random variable X (i.e. of μ_X)?

e) Find the probability density function ρ_Y and the distribution function F_Y of μ_Y, where μ_Y is defined as the "pull back" $\mu_Y := \mu \circ Y^{-1}$. F_Y is also called *distribution of* Y.

Problem 17: Maxwell-Boltzmann distribution in 2D (1+1+2+1+1+1)

Consider a particle with mass m in 2D whose speed $\mathbf{v} \in \mathbb{R}^2$ is a random variable distributed according to the probability density function $\rho_{\mathbf{v}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\rho_{\mathbf{v}}(\mathbf{v}) \propto \mathrm{e}^{-\beta E(|\mathbf{v}|)}, \tag{9}$$

where $\beta > 0$ is the inverse Temperature and $E(|\mathbf{v}|) = \frac{1}{2}m|\mathbf{v}|^2$ is the kinetic energy with mass $m > 0$.

a) Find $\rho_{\mathbf{v}}$ explicitly by determining the normalization constant.

b) Calculate the expectation value $\langle \mathbf{v} \rangle$ of $\mathbf{v}$.

As the energy E is a function of $\mathbf{v}$, it is itself a random variable.

c) Find ρ_E, the probability density function of the energy.
Hint: Check weather ρ_E is normalized to verify your result.

d) Comment on the fact that ρ_E exists even though energy, considered as a function from $\mathbb{R}^2$ to $\mathbb{R}$, is not invertible as required in Problem 15.

e) Calculate the expectation value $\langle E \rangle$ of the energy. What is hence the expectation value of the square of the speed?
Hint: The result is $\langle E \rangle = 1/\beta$.

f) Calculate the variance $\langle (E - \langle E \rangle)^2 \rangle$ of the energy.

Problem 18: Shannon information (1+2)

The Shannon information $H(p)$ of a probability vector $p \in [0,1]^n$ is given by

$$H(p) = -\sum_{j=1}^{n} p_j \log_2(p_j). \tag{10}$$

a) Show that H is well-defined, i.e., that $\lim_{x \searrow 0} x \log_2(x) = 0$.

b) Calculate the Shannon information of the following probability vectors

(i) $p = (1, 0, \ldots, 0) \in \mathbb{R}^n$
(ii) $p = (1, 1, \ldots, 1)/n \in \mathbb{R}^n$
(iii) $p = (1/2, 1/3, 1/6)$.